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Charge density waves and the replica method

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Abstract. — In this note I will present some analytic studies of the properties of charge density waves near the depinning point. Computations using the replica method are done in a simplified model.

1. Introduction.

The subject of this note will be the study of a simplified model for charged density waves [1-4] using the replica method. Before discussing the pinning of a real charge density wave in the simplified model it may be convenient to recall some of the properties of real charge density waves. Some possible justifications of the simplified model are presented.

2. Charge density waves.

In the model for charge density waves I consider here, the dynamical quantities are the fields \( \dot{\phi}_i(t) \), which satisfy the following differential equation [5]:

\[
\dot{\phi}_i = E + \Delta \phi_i - \cos(\phi_i - h_i),
\]

where \( \Delta \phi_i \) is the lattice Laplacian and \( i \) denotes a point of a \( d \) dimensional lattice. We are interested to know the time dependence of the current \( J(t) = \frac{1}{V} \sum_i \phi_i(t) \), when we start with a simple boundary condition at time equal to zero (e.g. \( \phi_i(0) = 0 \)).

For small values of \( E \), \( J \) goes to zero at large times, while for large values of \( E \) \( J \) goes to a non zero value at large times. We expect therefore the existence of a critical value of \( E \) (i.e. \( E_c \)) which separates the two regimes. It is reasonable to believe that, just at the critical point (i.e. \( E = E_c \)), the current decays as a power law:

\[
J(t) \sim t^{-\omega}
\]
One of the goals of a theory would be to compute the value of $\omega$, and more generally the behaviour of $J(t)$ at large times in the region where $E$ is near to $E_c$ [5]. This is a rather difficult task and we limit ourselves to the study of the properties of the model for $E$ less but near to $E_c$. In this case we do not need to consider the dynamical equations in detail, because we know that at large time $\phi$ will tend to a (stable) solution of the static equation:

$$0 = E + \Delta \phi_i - \cos (\phi_i - h_i).$$  \hfill (2.3)

If we consider a given sample in a volume $V$ we can define as $N(E, V)$ the number of solutions of equation (2.3). Usual arguments suggest that equation (2.3) should not have solutions for $E > E_c$ while it should have many solutions in the opposite regime where $E < E_c$. In other words it is convenient to define the function $f(E)$ as

$$f(E) = \lim_{N \to \infty} N(V, E) / E.$$ \hfill (2.4)

This function should be positive for $E > E_c$ and vanish just at the critical point.

In principle we could try also to compute the expectation value of any functional $A[\phi]$ defined as

$$\langle A[\phi] \rangle = \sum_{\phi = 1, N} A[\phi] / N,$$ \hfill (2.5)

where the sum runs over the $N$ solutions of equation (2.3).

However we cannot identify the expectation value in equation (2.5) with what we obtain when we start with a given configuration of $\phi$ at time zero, because in this case we select the solution of equation (2.3) which is mostly near to the initial condition (moreover only stable solutions would be relevant). However just because near the critical field the total number of solutions becomes small, we can conjecture that in the limit $E \to E_c$ the expectation value in equation (2.5) tends to the dynamic expectation value defined by solving equation (2.1) with fixed boundary conditions.

These arguments imply that the computation of $f(E)$ and of the expectation values in equation (2.5) would tell us not only the value of the critical point but also the properties of charge density waves just at the depinning point.

Using standard manipulations one should consider the partition function

$$Z^{(n)} = \int \prod_{a=1}^{n} d\phi_a d\psi_a d\lambda_a dh \exp \left( \int \left( i (\lambda_a (-\Delta \phi_a) + \cos (\phi_a - h) + E) + \psi_a (-\Delta + \sin (\phi_a - h) \psi_a) \right) \right),$$ \hfill (2.6)

where the index $a$ runs from 1 to $n$; $\psi_a$ and $\bar{\psi}_a$ are Fermionic (anticommuting) fields.

However the result is not quite correct because in this case one would sum over all the solutions with an extra factor equal to $\text{sign} \left( \det (-\Delta + \cos (\phi_a - h)) \right)$. This difficulty may be removed, as suggested in [7] by introducing $2m$ replicas of the $n$ Fermions; the final results are obtained by taking the analytic continuation to the point $m = 1/2$ starting from $m$ integer.

Similar formulae may be obtained if we start from an equivalent version of equation (2.3), i.e.

$$\sigma_i \arccos (E + \Delta \phi_i) = -\phi_i - h_i \mod (2\pi),$$ \hfill (2.7)

where the variables $\sigma_i$ are introduced in order to take into account the multivalueness of the function $\arccos$. 
Although this last case may lead to a simple integration over the variable $h$, a more careful study should be done in order to obtain some prediction on the models. It would certainly be interesting to firstly do the computation on a random Bethe lattice of fixed coordination number $z$. In the $z$ going to infinity limit one should obtain the same result of a simple minded field theory [6] (see Sect. 5 for a more precise definition of the model).

3. The coarse-grained model.

We can argue that the main effect of the fields $h_i$ is to pin the values of $\phi$ more or less strongly (strongly if the values of the $h$'s in a given region are similar, weakly if they differ of about $\pi$ so that their effects cancel ones with the others). We finally arrive at another model [1, 2] which is supposed to describe the large scale behaviour of charge density wave:

$$\dot{\phi}(t) = f(x) = \Delta \phi(t) + p + \eta,$$

where $\theta(x) = 0$ for $x < 0$ $\theta(x) = 1$ for $x > 0$, $\eta$ is a random uncorrelated Gaussian noise and $p$ is a control parameter which plays the same role as $E$. One can easily convince ourself that the critical value of $p$ is just $p = 0$.

Our task is now to compute the large time behaviour of $J(t)$ at $p = 0$. Let us assume that the previous equations have a solution in the continuum limit, where they become

$$\dot{\phi}(x, t) = f(x) = \Delta \phi(x, t) + p + \eta(x),$$

where the correlations among two $\eta$'s are given by

$$\eta(x)\eta(y) = \delta(x - y).$$

In this case we can use dimensional analysis. We can measure everything in dimensions of length ($[x] = 1$). If we do so we find:

$$[\eta] = -d/2 \quad [t] = 2 \quad [\phi] = (4 - d)/2 \quad [J] = -d/2.$$  \hspace{1cm} (3.4)

Dimensional analysis implies that

$$\omega = d/4.$$  \hspace{1cm} (3.5)

The crucial assumption is the existence of the solution of equation (3.2) in the continuum. In many cases stochastic differential equations with a dimensionless coupling are not well defined directly in the continuum limit. If a lattice spacing is introduced at a preliminary stage, one may find an additional dependence of the observables on the lattice spacing. Naive dimensional counting cannot be used and the renormalized operators (which have a finite expectation value in the continuum limit) have an effective dimension which differs from the naive dimension. The previous analysis assumes the absence of anomalous dimensions and it corresponds to a mean field approach.

The absence (or the presence) of divergences in the continuum limit should be studied using renormalization group techniques. This has not yet been done; however accurate numerical simulations in 1 and 2 dimensions show that the prediction $\omega = d/4$ is satisfied with an accuracy less than 1%, so that it seems rather reasonable (although we lack a formal proof) that equation (3.5) is an exact statement.
4. The replica study of the coarse-grained model.

In this section we are going to study the static solutions of the equations for the coarse-grained model, i.e.

\[ f \leq 0 \quad f = \Delta \phi_i(t) + p + \eta_i. \quad (4.1) \]

In this case the static equations are given by an inequality, not by an equality, so that the quantity that is interesting is the volume of the configurations of the field \( \phi \), which satisfy the inequality (4.1).

Therefore we can define the partition function for \( n \) replicas as

\[ Z^{(n)}(p) = \left( \int d[\phi] \theta(-f) \right)^n, \quad (4.2) \]

and the corresponding free energy density as

\[ F_n(p) = \lim_{V \to \infty} \frac{1}{nV} \ln \left( Z^{(n)} \right) \quad (4.3) \]

The logarithm of the most likely of the volume in \( \phi \) space is given by

\[ \mathcal{F}(p) = \lim_{n \to 0} F_n(p). \quad (4.4) \]

We expect that \( \mathcal{F}(p) \) goes to \(-\infty\) when \( p \to p_c^-\).

It is relatively easy to set up the computation of \( Z^{(n)}(p) \). We firstly introduce a lattice spacing in order to avoid divergences. Using the usual techniques, we get

\[ Z^{(n)} = \int d\eta \prod_{a=1, n} d[t_a] d[\lambda_a] d[\varphi_a] \exp(-F) \]

\[ F = i \int dx \lambda_a (\Delta \varphi_a + p - t_a + \eta), \quad (4.5) \]

where we have neglected a simple multiplicative factor (here the Jacobian is constant and Fermions are not necessary). The integral over the variable \( t \) goes from 0 to infinity. The integral over the field \( \varphi \) is trivial: it implies that \( \lambda \) is \( x \) independent; the previous formula simplifies to

\[ Z^{(n)} = \int d\eta \prod_{a=1, n} d[t_a] d[\lambda_a] d[\varphi_a] \exp(-F) = \int d\lambda_a \exp(-VF(p, \lambda)) \]

\[ \exp(-F(p, \lambda)) = \int_{-\infty}^{\infty} d\eta \int_0^{\infty} dt_a \exp \left( i \sum_{a=1, n} \lambda_a (t_a + \eta + p) - \eta^2/2 \right) = \]

\[ \prod_{a=1, n} 1/\lambda_a \exp \left( -\left( \sum_{a=1, n} \lambda_a \right)^2/2 \right) + i \sum_{a=1, n} \lambda_a + p. \quad (4.6) \]

In the limit where the volume \( V \) goes to infinity, the saddle point method may be used and it gives a saddle point at \( \lambda_a = -1/p \) for \( p \) negative. The corresponding expression for \( \mathcal{F}(p) \) is given by

\[ \mathcal{F}(p) = 1 + \ln(-p), \quad (4.7) \]
which diverges at \( p = 0 \), as was expected.

A simple computation shows that the correlation function of \( \phi \) in momentum space is given by \( 1/k^4 \), which agrees with the results of dimensional counting at \( p = 0 \).

We have seen that the approach suggested in the previous section can be used in this simplified case. The value of the threshold field and of the correlations near the critical point confirm the results coming from a naive analysis.

5. On the justification of the coarse-grained model.

In principle one could write a different form the coarse grained model, i.e.:

\[
\dot{\phi}(x, t) = f^\omega \theta(f),
\]

where \( f \) is the same as in equation (4.1). In other words \( \omega \) controls the response of a system at threshold. The choice \( \omega = 1 \), that we have done should be justified more carefully.

It would be natural to use for \( \omega \) the value coming from mean field theory. However in the infinite range charge density wave model, where a form of mean field theory is correct the value of \( \omega \) is equal to 1 or to \( 3/2 \) depending on the ratio in the Hamiltonian of the coefficients of the kinetic term and of the pinning term \( (\omega = 1 \text{ and } E_c = 0 \text{ in the weak pinning region, while } \omega = 3/2 \text{ and } E_c > 0 \text{ in the strong pinning region}) \).

In this situation we do not have an easy time to justify our choice \( \omega = 1 \). However mean field theory is very simplified in that it does not take into account the fact that the force of other spins on a single spin fluctuates with time. It seems reasonable that this sharp distinction between weak and strong pinning is an artefact of the infinite range model and it should disappear in finite dimensions where the threshold field is always different from zero.

Computations in finite dimensions are always difficult so it may be interesting to consider a model with infinite range, but finite coordination number \( z \). The equation of motion is

\[
\dot{\phi}_i = E + B/z \sum_k C_{i, k} (\phi_i - \phi_k) - \cos(\phi_i - h_i),
\]

where the matrix \( C \) is a random symmetric matrix, whose components are always 0 or 1, which satisfy the constraints

\[
\sum_k C_{i, k} = \sum_i C_{i, k} = z
\]

In other words the connection matrix \( C \) defines a random lattice in which each point has exactly coordination number \( z \). When \( z \) goes to infinity we recover mean field theory and the value \( B = 1 \) is the boundary among weak and strong coupling.

In this model each site feels a fluctuating field coming from the other points so that it is unclear if the analysis done in simple mean field theory still holds. In principle one should be able to control this kind of models with analytic arguments, but in practice it is not so easy. It is rather likely however that \( E_c \) is always different from zero and that \( \omega \) is independent of \( B \). The distinction between strong and weak pinning should be an artefact of the existence of an infinite number of neighbours in the naive mean field mode. Preliminary numerical simulations confirm this expectation: \( E_c \) is different from zero also in the weak pinning region. More careful computations are needed to compute the value of \( \omega \) and we hope that this problem can be soon clarified.
An analytic study of the dilute model would be very interesting and it could open the possibility of performing detailed computation in the finite dimensional models. This problem is rather difficult, however it seem to be not out of reach of present day theoretical techniques.

References