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Bifurcation in a random environment

Y. Pomeau

Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24 Rue Lhomond, 75231 Paris Cedex 05, France

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Abstract . — I investigate various instances of bifurcation in a random stationary environment. This is done for a model equation, with a bifurcation parameter fluctuating in space. The noise is either Gaussian or two states Poisson. I estimate then the order parameter as a function of a bifurcation parameter. This is mostly done by Lifshitz-tail type estimates on the linear part of the equation. However, near the threshold of instability in the absence of noise, this one-whatever small it is-becomes important and there the nonlinearity becomes crucial too.

One long standing interest of Rammal was the problem of localization in random media. In its classical formulation, this is a purely linear problem : one wants to know, given a random potential, the structure of the spectrum [1] and the properties of the eigenfunctions, if there are any, which is far from being granted. Some years ago, I suggested [2] to study the problem of nonlinear buckling of rectangular plates under longitudinal stress and with wavy boundaries. From localization theory, the first buckling mode should be localized, as are the eigenstates of the Schrödinger equation in a random potential. A somewhat idealised form of the same problem of bifurcation with random conditions was proposed recently by Zimmermann [3] and can be stated as follows. Consider the Swift-Hohenberg equation :

$$\left[\varepsilon - \left(\frac{d^2}{dx^2} + q_0^2 \right)^2 \right] a(x) = a^3(x) , \quad (1)$$

where $a(x)$ is a real function of x on the real line. The uniformly bounded solutions of (1) have a bifurcation at $\varepsilon = 0$ for $\varepsilon < 0$, the only solution of (1) is $a = 0$, although for $\varepsilon > 0$, a two parameter family of periodic solutions exist. One of the parameters in this family is a phase, irrelevant here, and the other the wavenumber, limited to an interval around q_0 . This model has a variational structure (as the equation of the buckling problem itself), as (1) is the Euler-Lagrange condition minimizing the functional :

$$V\{a\} = \int dx \left[\frac{\varepsilon a^2(x)}{2} - \frac{1}{2} \left\{ \left(\frac{d^2}{dx^2} + q_0^2 \right) a(x) \right\}^2 - \frac{1}{4} a^4(x) \right] , \quad (2)$$

thus it makes sense to single one solution out of all the ones of (1), that is the uniform bounded one with the highest potential $V\{a\}$. On an infinite line, this is not well defined, because the integral from $-$ to $+$ infinity in (2) is diverging. However we assume that we are looking at the solution with the maximum of V per unit length, even though this one requires to eliminate local defects altogether. From this optimal bifurcated solution, one can compute for instance the average absolute value of a , $\langle |a(\cdot)| \rangle$, the averaging being by gliding over x . This quantity $\langle |a(\cdot)| \rangle$ is a continuous function of ε , that is equal to 0 for ε negative and, for ε positive expands near zero as :

$$\langle |a(\cdot)| \rangle = \frac{2\varepsilon^{1/2}}{3^{1/2}} + \frac{\varepsilon^{3/2}}{3^{1/2} \cdot 256} + \dots$$

Let us formulate our problem of bifurcation in a random environment as follows : we use the same equation as (1), but assume now that ε is a random function of space : $\varepsilon(x)$. To make easier the discussion, we shall assume this random function to be the sum of a given random function $\eta(x)$, plus one bifurcation parameter u , independent of x , so that (1) becomes :

$$\left[u + \eta(x) - \left(\frac{d^2}{dx^2} + q_0^2 \right) \right] a(x) = a^3(x), \quad (3)$$

Replacing $\frac{\varepsilon a^2(x)}{2}$ in (2) by $\frac{(\eta + u)a^2(x)}{2}$, one gets the energy associated to (3) :

$$V\{a\} = \int dx \left[\frac{(\eta + u)a^2(x)}{2} - \frac{1}{2} \left\{ \left(\frac{d^2}{dx^2} + q_0^2 \right) a(x) \right\}^2 - \frac{1}{4} a^4(x) \right], \quad (4)$$

so that we can ask ourselves the question of the behaviour of $\langle |a(\cdot)| \rangle$ as a function of u , η being a given random function of x . Actually we shall be concerned with the behaviour of $\langle |a(\cdot)| \rangle$ for the optimal $a(x)$, i.e., the one extremalizing the energy. Indeed this depends on the random function $\eta(\cdot)$. We shall consider three possible choices of random function. First $\eta(\cdot)$ Gaussian with a coloured noise, then $\eta(\cdot)$ Gaussian with white noise, and finally $\eta(\cdot)$ Poissonian.

1. $\eta(\cdot)$ Gaussian coloured noise.

In that case, $\eta(\cdot)$ is a Gaussian Random function of x specified by its average value $\langle \eta(\cdot) \rangle = 0$, as well as by the correlation $\langle \eta(x)\eta(x') \rangle = \phi(|x - x'|)$, $\phi(\cdot)$ being a smooth function of its argument tending to zero at infinity with a range λ . We want to compute the behaviour of $\langle |a(\cdot)| \rangle$ as the bifurcation parameter u in (3) tends to minus infinity. In this domain, the integrand of the energy is almost certainly zero, unless the first term, that is $\frac{(\eta(x) + u)a^2(x)}{2}$ manages to become positive. This happens somewhere, because a random Gaussian function $\eta(\cdot)$ has always a probability, whatever small, to reach a large positive value, that will overbalance the large negative u .

An interval of coherence of $\eta(\cdot)$ has a length λ , and in this interval the probability that η is everywhere larger than $|u|$ is about $\exp(-\beta u^2)$, where β^{-1} is approximately $\frac{1}{\lambda} \int dx \phi(x)$. In this interval, a local instability will begin to grow at the value of u under consideration. As u changes, by increasing for instance but staying large negative, the value of $\langle |u(\cdot)| \rangle$ will remain dominated by the contribution of the intervals of the real line that have just bifurcated in this sense, because the ones that have bifurcated for still lower values of u are too rare to contribute

significantly : their amplitude have grown algebraically as a function of u but they are rarer by an exponential term. From all of this one deduces that $\langle |u(\cdot)| \rangle$ behaves as $\exp(-\beta u^2)$ as u tends to minus infinity. There is likely a multiplicative factor depending on u like a power, that cannot be attained by this analysis.

2. $\eta(\cdot)$ Gaussian white noise.

The calculation in that case follows lines very similar to the previous case, but for one supplementary complication arising from the fact that $\eta(\cdot)$ has zero range correlations. We shall estimate first the probability that in an interval of length Λ the average value of $\eta(\cdot)$ is bigger than some fixed value U . This average value is

$E_\Lambda = \frac{1}{\Lambda} \int_\Lambda dx \eta(x)$, and it is a Gaussian random variable itself, being linearly related to a Gaussian quantity. Let us specify the distribution of $\eta(\cdot)$ by $\langle \eta(\cdot) \rangle = 0$ and $\langle \eta(x)\eta(x') \rangle = \omega \delta(x - x')$, then the probability distribution for E_Λ is the Gaussian

$$P(E_\Lambda) = \left(\frac{\Lambda}{2\omega\pi} \right)^{1/2} \exp \left(-\frac{\Lambda E_\Lambda^2}{2\omega} \right),$$

whence a domain of length Λ will be unstable against the growth of a fluctuation of $a(\cdot)$ if in this domain the value of E_Λ is larger than $-u$, that will occur with a probability $\exp \left(-\frac{\Lambda u^2}{2\omega} \right)$. The gravest mode of the linearized equation in this domain will have a wavelength Λ , much smaller a priori than 1 : otherwise the probability would be too small, because of the exponential form of $P(E_\Lambda)$. On the other hand, this length cannot be too small, otherwise the dominant term in the equation (1) would be the fourth derivative with respect to x . This means that the optimal Λ is when this fourth derivative term is of the same order as the other terms in (1), that implies $\frac{1}{\Lambda^4}$ of order $|u|$, or $\Lambda \sim |u|^{-1/4}$ so that the probability of having a bifurcation to a nonzero $a(\cdot)$ in a domain with this length is $\exp \left(-\frac{Cu^{7/4}}{\omega} \right)$, where C is a numerical constant. This last exponential yields, as in the previous case the dominant order term (again up to algebraic prefactors) to $\langle |a| \rangle$ in the $u \rightarrow -\infty$ limit.

3. Bifurcation threshold in the small Gaussian noise limit.

Till now we have presented results depending essentially on the linear part of the Swift-Hohenberg equation, and that could be rather easily reformulated as a Lifshitz-tail analysis of the linear part of this equation (see at the end for possibility of a more detailed analysis). Now we are going to consider a range of values of parameters where the nonlinear part of (1) plays a crucial rôle, that is the neighborhood (in a sense to be precised) of $u = 0$, and in the limit of a small amplitude Gaussian noise.

Near the threshold, the relevant equation in the Newell-Whitehead-Segel [4] approximation of (1), that is an envelope equation for the complex amplitude of fluctuation $a(\cdot)$ with a phase factor $\exp(iq_0 x)$. Let $A(x)$ be this complex amplitude, a slowly varying function of x thus, that is the solution of :

$$[u + \eta(x)]A(x) + 4q_0^2 \frac{d^2 A}{dx^2} - \frac{3}{4}|A|^2 A = 0, \quad (5)$$

the validity of this implying that u and $\varepsilon(\cdot)$ are small and that the long range part of $\eta(\cdot)$ only is kept ("long range" meaning over a length scale much longer than the wavelength $\frac{1}{q_0}$). I shall restrict myself to the case of a Gaussian white noise for $\eta(\cdot)$, the case of a coloured noise being very much the same. That η is small implies that ω is small. From the reasoning we already presented, the order of magnitude of the average of η over a length Λ is $(\omega/\Lambda)^{1/2}$, so that I shall consider this as the order of magnitude of the $\eta(x)$ -term in (5). Let take first the bifurcation parameter u as zero in (5). The length Λ on average has to make a compromise : if it is too large, the average value of η is too small and the amplitude of $A(\cdot)$, being of order $|\eta|$ becomes too small too, although if this length becomes too small, the second derivative in (5) becomes dominant and the optimal solution is zero. Then the order of magnitude of Λ is determined by the balance of the three terms in (5), given that $\eta(\cdot)$ is of order $(\omega/\Lambda)^{1/2}$. This yields $\Lambda \sim \omega^{-1/3}$ and $A \sim \omega^{1/3}$. Moreover the range of values of u where this applies is such that $u \sim \eta \sim (\omega/\Lambda)^{1/2} \sim \omega^{2/3}$. Otherwise, either the dominant term in (1) is $(ua(x))$, if u is positive, or one has to rely upon transcendental estimates if u is negative.

4. Poisson noise near threshold.

By Poisson noise, I mean a two valued noise $\eta(\cdot)$ switching randomly from $-c$, c positive, to 0, as x changes. Considering x as a time variable, this is Poissonian when the probability of switching is constant per unit "time". This kind of noise has properties rather different from a Gaussian noise, in particular there is obviously no large values of $|\eta(\cdot)|$, that is bounded by definition. This implies in particular that the average $\langle |a| \rangle$ is zero for u less than c . For u bigger than c , the sum $u + \eta(x)$ becomes eventually positive, but over length typically of order λ , where λ is the persistence "time" of the Poisson process. Let us investigate the threshold $u = c + \delta$, δ small positive. As in the previous case, because we are near threshold, the relevant equation is the amplitude equation (5) limit, for the full equation (1). If λ was infinite, one would be back to the standard bifurcation problem for the amplitude equation on the infinite line, and one would have $\langle |a| \rangle \sim \delta^{1/2}$ near threshold, as usual. But in the present case this is not what happens, because the amplitude equation associates a length scale to an amplitude scale : on a segment of length L , the bifurcation takes place at $\delta = \frac{\pi^2}{L^2}$, according to a standard result. Hence for a Poissonian $\eta(x)$, the bifurcation will occur first in the longest intervals, but these intervals have a probability of occurrence decreasing exponentially when their length increases, like $\exp\left(-\frac{L}{\lambda}\right)$. Furthermore this length L is associated with a bifurcation threshold $\delta = \frac{\pi^2}{L^2}$, whence the estimate of the relative probability along x of being in a bifurcated interval : $\exp\left(-\frac{\pi\delta^{-1/2}}{\lambda}\right)$, which gives, again up to algebraic subdominant factors, the law of growth of $\langle |a| \rangle$ near $\delta = 0|_+$. Notice that in this case, I did as if the various bifurcated intervals were independent, which is rather natural, since they are very rare near threshold and so very far away from each other. Thus the overlap of bifurcated solutions in different intervals may be neglected, as we did.

5. Final remarks.

I have presented here an heuristic approach of the question of bifurcations in a nonlinear environment, as posed by the Swift-Hohenberg equation. One might wonder if a more analytical

approach is feasible. This is likely to be true at least in the case of Gaussian white noise. In that case, one may write the fourth order equation (1) as a set of four coupled nonlinear differential equations of first order :

$a_0 = a(x)$, $a_1 = \frac{da_0}{dx}$, $a_2 = \frac{da_1}{dx}$, $a_3 = \frac{da_2}{dx}$, and
 $\frac{da_3}{dx} = v(\mathbf{a}) + \varepsilon(x)a_0$, where \mathbf{a} is the vector (a_0, a_1, a_2, a_3) and where $v(\mathbf{a}) = (v - q_0^4)a_0 - a_0^3 - 2q_0^2a_2$. For such a white Gaussian noise, one applies the Chapman-Kolmogoroff method to get the equation for the evolution of the probability distribution of \mathbf{a} , let $P(\mathbf{a}, x)$ be this probability. It obeys the Chapman-Kolmogoroff equation :

$$\frac{\partial P}{\partial x} + \frac{\partial (a_1 P)}{\partial a_0} + \frac{\partial (a_2 P)}{\partial a_1} + \frac{\partial (a_3 P)}{\partial a_2} + \frac{\partial (v(\mathbf{a}) P)}{\partial a_3} = \frac{\omega \partial}{2 \partial a_3} a_0^2 \frac{\partial P}{\partial a_3}, \quad (6).$$

The results presented before concern the equilibrium solution of this equation $\left(\frac{\partial P}{\partial x} = 0\right)$. I plan to come to this in the future. It is worth noting however that the general stationary solution of this equation in the absence of noise is a rather arbitrary (but integrable and positive indeed) and of the form $P(H)$, where $H(\mathbf{a})$ is the constant of the motion of the dynamical equation for \mathbf{a} in the absence of noise, that was discovered by Zaleski [5] and that reads :

$$\begin{aligned} H(\mathbf{a}) &= (\varepsilon - q_0^4) \frac{a^2}{2} - \frac{a^4}{4} - q_0^2 \left(\frac{da}{dx}\right)^2 - \frac{da}{dx} \frac{d^3a}{dx^3} + \frac{1}{2} \left(\frac{d^2a}{dx^2}\right)^2 \\ &= (\varepsilon - q_0^4) \frac{a_0^2}{2} - \frac{a_0^4}{4} - q_0^2 a_1^2 - a_1 a_3 + \frac{1}{2} a_2^2. \end{aligned}$$

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