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Mean-field solution of a block-spring model of earthquakes

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Abstract. — A mean field version of the Burridge-Knopoff block-spring stick-slip model of earthquake faults is mapped onto a cycled generalization of the democratic fiber bundle model (DFM). This provides an exactly soluble model which describes the set of earthquakes preceding a major earthquake. We find the coexistence of 1) a differential Gutenberg-Richter distribution \( d(\Delta) \sim \Delta^{-5/2} \) of bursts of size \( \Delta \), with a cut-off \( \Delta_{\max} \sim (\sigma_r - \sigma)^{-1} \) as the stress \( \sigma \to \sigma_r \), and 2) a run away occurring at a well-defined stress threshold \( \sigma_r \). The total number of bursts of size \( \Delta \) up to the run away scales as \( D(\Delta) \sim \Delta^{-5/2} \). The exponent 5/2 reflects the occurrence of larger and larger events when approaching the run away instability (Omori's law for foreshocks). The Gutenberg-Richter and Omori power laws are not associated with a stationary criticality but to fluctuations accompanying the nucleation of the run away. Introducing long range correlations in the model lead to a continuous dependence of the above exponents as a function of the correlation exponent.

Understanding earthquake dynamics and plate tectonic mechanics is now the focus of an important research effort in the physical community [1-9], since it has been realized that they provide one of the most interesting physical realization of spatio-temporal complex dynamics [10], a subject of large current interest which covers many different fields [9, 11]. Beginning with the Burridge-Knopoff model [12], a variety of increasingly sophisticated mechanical models that simulate the occurrence of earthquakes have been studied [3-8]. These characteristically produce power laws for the rate of occurrence of model earthquakes and seem to display the property of self-organized criticality (SOC) [13]. At present, there is no real theoretical understanding why such power laws appear in block-spring stick-slip Burridge-Knopoff type models or even in its various cellular automata versions.

Consider the Burridge-Knopoff block-spring model of earthquake faults consisting in an array of blocks of mass \( m \) coupled to each other by harmonic springs of strength \( k_c \) and pulled to the right by an upper rigid surface acting through pulling springs of strength \( k_p \). The blocks are in contact with a fixed rough substrate exerting a static friction force \( F_s \) on them. The blocks have initial random positions and are motionless. The upper pulling surface steadily increases its displacement and thus the force exerted on each block increases.

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At some point, the total force exerted on a block, which is the sum of the pulling force and the action of the block neighbors, overcomes $F_s$ and this block begins to slide. This sliding may either stop eventually or trigger the sliding of its neighbors, possibly cascading to produce a large event. The difficulty of solving this model theoretically comes from the interplay between 1) the non-linear threshold behavior of the friction law, 2) the fact that the event at time $t$ depends on all the previous history of past events and 3) the local coupling between neighboring blocks, which puts this model in the large class of $N$-body spatio-temporal dynamical models. Inspired by the standard treatment of phase transitions, we now propose a mean field (MF) version of this model which simplifies the above complexity and allows for an exact solution.

Mean field model.

The MF treatment of the paramagnetic-ferromagnetic phase transition amounts to replacing the action of neighboring spins on a given spin by an effective average magnetic field, which is self-consistently calculated. The difficulty is thus reduced from a $N$-body problem to a 1-body problem. The MF version of the Burridge-Knopoff model amounts to replacing the forces exerted by the neighboring blocks on a given block, acting through the springs $k_p$, by an average effective force. Consider a given initial random static block configuration defined by the set of positive deviations $\{x_n, n = 1 \text{ to } N \}$ from the position at which zero force is exerted. This set of deviations defines a set of force thresholds for sliding given by $\{X_n = F_s - k_p x_n, n = 1 \text{ to } N \}$. As the upper surface begins to move by a displacement $x(t)$, the incremental force exerted on each block is $\sigma = k_p x(t)$. When $\sigma$ becomes larger than $\text{Min}_n \{X_n = F_s - k_p x_n, n = 1 \text{ to } N \}$, blocks begin to slide. In the spirit of cellular automata modeling, we assume, in addition, that once a block has stopped after it has slid some distance, it is unable to provide any contribution against the pulling action of the upper surface. This takes into account the fact that a block usually slides ahead of the pulling surface [3]. We also assume that blocks, which have slid once, are not allowed to slide again in a given cycle and do not exert any force on the pulling surface. Thus, the total force $F = N \sigma$ must now be shared among all remaining blocks which have not slid. If $k$ blocks have slid (or « failed »), the resulting force on the remaining blocks is now $\sigma N/(N - k)$ per block. As the upper surface moves further, $\sigma$ increases and an increasing number of blocks slide until all the blocks eventually slide in a runaway event. To ensure ongoing motion after all blocks have slid once, we then assume that the dynamical sliding events on each block have placed them in a new random set of positive deviations $\{x_n, n = 1 \text{ to } N \}$. The sliding process starts again and can then go on infinitely, cycle after cycle. The assumption that blocks which slid once never exert any force is reasonable when considering the set of earthquakes which precede a major one. After a run-away occurs, all blocks are assumed to exert a force which depends on the specific block positions.

This model, when restricted to a single cycle, is identical to a well-known model of rupture in random media, dubbed the « democratic fiber bundle model » (DFB) [14-20], which has been introduced initially to describe the rupture properties of long flexible cables or low-twist yarns. It is made of $N$ independent parallel vertical fibers with identical spring constant $k_p$ and identically distributed independent random failure thresholds $X_n, n = 1, \ldots, N$, distributed according to the cumulative probability distribution $P(X_n < x) = P(x)$. A total force $F$ is applied to the system and is shared democratically among the $N$ fibers. As $F$ increases, more and more fibers break down until the complete final rupture. The correspondence of the two models is that the rupture of a fiber corresponds to the sliding of a block. In the Burridge-Knopoff model, the disorder on the failure thresholds $X_n$ can be viewed
as a snap-shot of the dynamical evolution of the block positions $x_n$ through the relationship $X_n = F_s - k_p x_n$. In essence, the present MF model attempts to describe the chaotic dynamics by a stochastic description.

Some properties of the DFB model in the continuous limit.

Let us first recall briefly the main known properties of the DFB model. Let us note $X_1, X_2, \ldots, X_N$, the ordered increasing sequence of strengths of the individual fibers. Under a total load $F$, a fraction $P (\sigma = F/N)$ of the threads will be submitted to more than their rated strength and will fail immediately. After this first step, the total load will be redistributed by the transfer of stress from the broken links to the other unbroken links. This transfer will in general induce secondary failures which in turn induce tertiary ruptures and so on. The properties of this rupture problem is obtained by noting that the total bundle will not break under a load $F$ if there are $n$ links in the bundle each of which can withstand $F/n$. In other words, if the first $k - 1$ weakest links are removed, the bundle will resist under a force smaller than or equal to

$$F_k = (N - k + 1) X_k$$

since there remain $(N - k + 1)$ links of breaking strength larger than or equal to $X_k$. The strength $F_N$ of the bundle is then given by $F_N = \max \{ (N - k + 1) X_k : 1 \leq k \leq N \}$ which goes to $N \max_x \{ x(1 - P(x)) \}$ in the limit $N \to +\infty$. It can be shown [15] that $F_N$ obeys a central limit theorem according to which the probability that the global failure threshold $F_N$ be equal to $F$ is $P (F_n = F) \sim (2 \pi N)^{-1/2} \exp \{- (F - N \theta)^2 / 2 N x_0^2 \}$, and thus $F_N$ converges to $N \theta$ where $\theta = xo (1 - P(x_0))$ is the unique maximum of $x(1 - P(x))$ at $x = x_0$. The number $k(\sigma)$ of links which have failed under the force $F = N \sigma$ is [19]

$$k(\sigma) = NP (x(\sigma)),$$

where $x(\sigma)$ is defined by

$$\sigma = x(\sigma) \{ 1 - P (x(\sigma)) \}.$$

The number of remaining links is therefore $N [1 - P (x(\sigma))] = F / x(\sigma)$ and $x(\sigma)$ is thus the force exerted on each surviving fiber. Consequently, the stress ($\sigma$)-strain ($\varepsilon$) characteristic of the system is

$$x(\sigma) = k_p \varepsilon(\sigma).$$

Just before complete failure of the bundle, the total number of broken links is $k_n = NP (x_0)$. The remaining $N [1 - P (x_0)]$ fibers break down suddenly in one sweeping runaway event when $\sigma \to \theta$ i.e. $x(\sigma) \to x_0$. For $F \leq F_N$, i.e. $\sigma \leq \theta$, $x(\sigma)$ is in the neighborhood of $x_0$ and can be expressed under the form

$$x(\sigma) - x_0 = -A (\theta - \sigma)^{1/2} \quad \text{for} \quad \sigma \to \theta,$$

where $A$ is a coefficient which depends upon the cumulative distribution $P(x)$. The stress-strain characteristics $\sigma(\varepsilon)$ is thus quadratic near global rupture with a vanishing slope at $\sigma = \theta$ [19].

The number of links which have failed under the stress $F$ is given by

$$k(\sigma)/N = P (x_0) - B \{ \theta - \sigma \}^{1/2} \quad \text{for} \quad \sigma \to \theta,$$
where $B$ depends upon the cumulative distribution $P(x)$. Expression (6) implies a very rapid increase of the number of broken links as $\sigma \to \theta$ for which $k(\sigma)$ tends to $NP(x_0)$ with a slope exhibiting a square root singularity.

These laws imply an interesting dependence of the rate of elastic energy release as the run away event is approached. For a given applied stress $\sigma$, the elastic energy stored in the bundle of surviving fibers is $E(\sigma) = \epsilon(\sigma) x(\sigma) k(\sigma)$, where $\epsilon(\sigma)$ is given by equation (4), $x(\sigma)$ by equations (3) and (5) and $k(\sigma)$ by equations (2) and (6). Using equations (3-4), we obtain

$$E(\sigma) = N \epsilon [k_p \epsilon - \sigma] .$$

(7)

If the system is driven at a constant strain rate $d\epsilon/dt$, $dE/dt$ goes to a constant as $\sigma \to \theta$. If, on the other hand, the system is driven at a constant stress rate, the rate $dE/dt$ of elastic energy release diverges when approaching global failure according to $dE/dt \sim (\theta - \sigma)^{-1/2}$. This corresponds to a marked average increase of rupture activity prior to the run away analogous to Omori’s law for foreshocks [21]. These results also underline the sensitivity of the behavior with respect to the loading path. In the models studied in reference [3] for instance, a constant strain rate was chosen. It is not clear whether this is the case in nature since a given fault is surrounded by an elastic medium deteriorated by many other faults which interact, leading to ill-defined boundary conditions. Therefore, the loading path of a real fault is probably intermediate between the pure constant strain or stress rate, which may explain why increased foreshock activity is not always observed before a main large earthquake.

Burst properties of the DFB model due to the discrete nature of the elements (fibers or blocks).

When looking at the fiber (block) scale, simultaneous failure of many fibers can occur due to « small scale » fluctuations of the strength of bundle subsets. Indeed, the sequence $\{F_k\}$ of external loads given by equation (1) at which the fibers would fail do not form a monotonically increasing sequence [20] (see Fig. 1). The random variables $X_k$ are indeed put in increasing order but they are multiplied by a monotonically decreasing factor $(N + 1 - k)$ as $k$ increases.

On the average, we have $P(X_k) = k/N$ which implies that

$$\langle X_{k+1} - X_k \rangle = [dP/dx|_{x_k}]^{-1} N^{-1}$$

and

$$\langle F_{k+1} - F_k \rangle = [1 - P(X_k)] [dP/dx|_{x_k}]^{-1} - X_k = [dP/dx|_{x_k}]^{-1} d \{x[1 - P(x)]\}/dx|_{x_k} .$$

(8)

Note that $\langle F_{k+1} - F_k \rangle$ is positive as long as $X_k < x_0$ and vanishes on approaching the run away event as

$$\langle F_{k+1} - F_k \rangle \sim (x_0 - x) .$$

(9)

The fluctuations of $F_k$ are given by $\langle [F_{k+1} - F_k]^2 \rangle \sim \langle [X_{k+1} - X_k]^2 \rangle - \langle X_{k+1} - X_k \rangle^2$ which is finite and weakly dependent of $x$. Starting from a stable configuration corresponding to some value $F_k$, a simultaneous rupture of $\Delta$ fibers, which can be called an event or burst of size $\Delta$, occurs if $F_n < F_k$ for $k + 1 \leq n \leq k + \Delta$ and $F_{k+\Delta+1} \geq F_k$. Since the set of failure
Fig. 1. — Strength $F_k$ of a bundle of $N$ fibers as a function of the number $k$ of broken fibers; (below): magnified view of the dependence of $F_k$ showing the random walk in the space of forces $F_k$, the role of time being played by $k$.

thresholds $X_k$ are independent random variables, the function $F_k$ undergoes a random walk where the displacement is $F_k$ and the time is $k$ (see Fig. 1). This random walk is biased by a drift equal to $\langle F_{k+1} - F_k \rangle$ which vanishes at the instability threshold (Eq. (9)). The bias is large far away from the instability threshold and only small bursts occur. As $x \to x_0$, $\langle F_{k+1} - F_k \rangle \to 0$ and the fluctuations completely dominate the burst occurrence.

In the random walk picture, it is easy to obtain the probability that a burst of size $\Delta$ occurs after $k$ fibers have been broken, i.e. at some value $F_k$. This corresponds to the probability $p_1(\Delta)$ of first return to the origin of a random walker (see Fig. 1). In the absence of any bias, the probability to be found at the origin after $\Delta$ steps decays as $\Delta^{-1/2}$ and the probability to return for the first time at the origin is $p_1(\Delta) \sim \Delta^{-3/2}$ [22]. Thus, the local differential distribution $d(\Delta)$ of bursts of size $\Delta$ is given by

$$d(\Delta) \sim p_1(\Delta) \sim \Delta^{-3/2}$$

This law (10) holds for $\Delta \leq \Delta_{\text{max}}(x)$, where $[\Delta_{\text{max}}(x)]^{1/2} \sim (x_0 - x) \Delta_{\text{max}}(x)$, is such that the drift $(x_0 - x)$ brings a displacement smaller than the typical random walk excursion in the time $\Delta_{\text{max}}(x)$:

$$\Delta_{\text{max}}(x) \sim (x_0 - x)^{-2} \quad \text{for} \quad x \to x_0.$$  (11)
Equation (11) implies that an event of size $\Delta$ may occur only when $\Delta_{\text{max}}(x) \geq \Delta$, i.e. $x_0 - x \leq \Delta^{-1/2}$. The powerlaw (11) is reminiscent of the Omori's law for foreshocks often observed for real earthquakes [21]: $\Delta_{\text{max}}(x) \sim (\theta - \sigma)^{-1}$, using equation (5).

Let us now calculate the total number $D(\Delta)$ of bursts of size $\Delta$. $D(\Delta)$ is the product of the number $n(\Delta)$ of bursts which can be larger than $\Delta$ by the probability $p_1(\Delta)$ that a burst is of size $\Delta$. To determine $n(\Delta)$, we note that on average, $dk \sim \langle \Delta \rangle(x) \, dn$, where $dk$ is the number of fibers broken in $dn$ bursts. We do not have a strict equality because a finite fraction of the fibers breaks down in elementary bursts of size unity. The average size $\langle \Delta \rangle(x)$ of a burst is given by

$$\langle \Delta \rangle(x) = \int_{\Delta_{\text{max}}(x)}^{\Delta} \Delta \, p_1(\Delta) \, d\Delta \sim \left[\Delta_{\text{max}}(x)\right]^{1/2} \sim (x_0 - x)^{-1} \sim (\theta - \sigma)^{-1/2},$$

using equation (5). Using equations (5) and (6) which show that $dk \sim N \, dx$ for $(x_0 - x) \ll 1$, we have $dn \sim N(x_0 - x) \, dx$, which allows to obtain

$$D(\Delta) \sim p_1(\Delta) \int_{x_0 - \Delta^{-1/2}}^{x_0} \Delta_{\text{max}}(x) \, (dn/dx) \, dx \sim p_1(\Delta) \, \Delta \sim \Delta^{-5/2} \quad (12)$$

The weighting factor $\Delta^{-1}$ in $D(\Delta)$ results from two ingredients: 1) due to the cut-off $\Delta_{\text{max}}(x)$ introduced by the bias, bursts of size larger or equal to $\Delta$ occur only when $x$ is sufficiently close to $x_0$; 2) as $x \to x_0$, there are fewer and fewer bursts since they are larger and larger.

This result (12) has been previously derived [20] using a detailed probability analysis and has been checked numerically. The present simple derivation in terms of biased random walks clarifies its physical origin and the fundamental reason for the universality of the MF exponents. Furthermore, it shows the coexistence of 1) a differential local distribution of bursts given by equation (10) with an exponent $3/2$ and 2) the total number of bursts of size $\Delta$ given by equation (12) with a larger exponent $5/2$. Applied to earthquakes, this result suggests that there is no contradiction in observing a small $\ll b$-value $= 1/2$ in a restricted time interval and a larger $\ll b$-value $= 3/2$ when the time interval is extended up to the occurrence of a great earthquake. Our result may provide a clue for the observed drift of $b$-values often observed before an impending earthquake. Note that the large $\ll b$-value $= 3/2$ which is predicted here for the cumulative distribution of earthquakes up to the largest run-away is in good quantitative agreement with the recent study [22, 23], which indicates that $0.8 \ll b \ll 1.2$ for small earthquakes and $b = 1.5$ for larger earthquakes for which the rupture dimension is larger than the downdip width of the seismogenic layer.

Effect of long-range correlations.

The main difference between our MF DFB model and the original Burridge-Knopoff stick-slip model is the existence of additional correlations in the latter, which are dynamically constructed from the action of the local couplings between the blocks. It is possible to obtain a feeling for the effect of such correlations by introducing in the MF model a correlation in the set of failure thresholds $X_k$ and therefore in the $\{F_k\}$ sequence. It is assumed that this correlation reflects the one which is built up dynamically in the complete Burridge-Knopoff block model. This procedure is thus an attempt to go beyond mean field treatment. However, the conclusions must be limited since no procedure is given to fix the value of the correlation exponent, which is kept as an arbitrary input parameter.

We thus assume that the $F_k$'s are correlated such that the correlation function $C(n) = \langle F_k \, F_{k+n} \rangle - \langle F_k \rangle \, \langle F_{k+n} \rangle \sim n^{-\gamma}$ is long ranged. Consequently, the standard devi-
atation $\mathcal{C} = \langle F^2 \rangle^{1/2}$ scales as $\mathcal{C} \sim \Delta^y$ with a superdiffusive exponent $1/2 < \nu = 1 - y/2 < 1$ for $0 \leq y < 1$ and $\nu = 1/2$ for $y > 1$ [22]. Correlations such that $y > 1$ do not lead to any modifications to the previous analysis. We thus restrict our attention to the case $0 \leq y \leq 1$.

The probability to return for the first time at the origin after $\Delta$ steps is modified and reads [24] $p_1(\Delta) \sim \Delta^{-(2 - \nu)} \sim \Delta^{-(1 + y/2)}$. Using this new powerlaw dependence, we then follow step by step the same reasoning as described above. $\Delta_{\text{max}}$ is such that $[\Delta_{\text{max}}(x)]^y \sim (x_0 - x) \Delta_{\text{max}}(x)$, which yields $\Delta_{\text{max}}(x) \sim (x_0 - x)^{-1/(1 - \nu)}$. The average burst size $\langle \Delta \rangle(x)$ is given by $\langle \Delta \rangle(x) = [\Delta_{\text{max}}(x)]^y \sim (x_0 - x)^{-y/(1 - \nu)}$. We thus finally get

$$D(\Delta) \sim p_1(\Delta) \int_1^{\Delta_{\text{max}}(x)} (dn/dx) \, dx, \quad \text{i.e.}$$

$$D(\Delta) \sim p_1(\Delta) \gamma(\Delta) \sim \Delta^{-(3 - \nu)} \sim \Delta^{-(2 + y/2)} \quad \text{for} \quad 0 \leq y \leq 1.$$  \hspace{1cm} (13)

Expression (13) exhibits a power law behavior with an exponent showing a continuous dependence on the rupture threshold correlation exponent for $0 \leq y \leq 1$. It is interesting to note that the limiting case $y \to 0$ yields a "b-value" $b = 1$ (defined as the exponent of the integral of $D(\Delta)$ from 1 to $\Delta$), which is the value determined numerically in the Burridge-Knopoff block-spring models for the distribution of earthquake sizes [3]. According to the present theory, it is tempting to interpret this value $b = 1$ as the signature of the longest possible correlation range of the block positions. Our analysis provides another way to look at the non-universal behavior of the Gutenberg-Richter exponent, as being dependent on the level of correlation. This concept should be compared with a recent numerical study of a quasi-static two-dimensional cellular automaton version of the Burridge-Knopoff spring-block model of earthquakes [8], according to which the exponent was found to depend on the level of conservation in the cellular automaton. The relation between the level of conservation [8] and the degree of correlation suggested here remains to be clarified.

Concluding remarks.

The "democratic fiber bundle" MF approximation of the Burridge-Knopoff block-spring model of earthquakes has been shown to exhibit various power law analogous to the famous Gutenberg-Richter [10, 25] and Omori [21] laws for earthquake occurrence. In this MF model, these power laws are not associated with criticality in a pure sense but describe the fluctuations prior to the nucleation of the run away event. Indeed, there is a close analogy between the burst fluctuations observed before the occurrence of the run away event and the droplet fluctuations before the nucleation of the critical droplet which destabilizes into the new phase in a standard first order phase transition [26]. The exponent $5/2$ found here for the distribution $D(\Delta)$ is also characteristic of mean field percolation and spinodals. We note that the powerlaw distribution of events (see Eqs. (10, 12, 13)) reflects the discrete nature of the model, i.e. is associated with the existence of a microscopic block scale. This small scale cut-off is, however, of no importance for the continuous properties of the deformation since a central limit theorem holds.

The exponents of the burst size distributions are universal which implies that the results are insensitive to the details of the distribution of block positions after each run away event. This result justifies our assumption which ensures continuing motion, corresponding to the so-called "seismic cycle" model for the recurrence of great earthquakes [25], that each run away event amounts to a reshuffle of the breaking thresholds for the next dynamical process. Our analysis shows that there is no contradiction in observing power laws in the distribution of earthquakes between run away events and the idea of a seismic cycle. Of course, it is quite
possible that, in reality, the power law extends down to the largest earthquakes [1, 2, 4c-d, 5] and that our results are specific to the MF model.

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