The Domany-Kinzel cellular automaton revisited

G. Kohring, M. Schreckenberg

Classification
Physics Abstracts
05.70F — 02.50 — 89.80

Short Communication

The Domany-Kinzel cellular automaton revisited

G.A. Kohring and M. Schreckenberg

Institut für Theoretische Physik, Universität zu Köln, Zülpicher Straße 77, D–5000 Köln, Germany

Abstract. — New analytic and numerical results for the one-dimensional Domany-Kinzel cellular automaton are presented. Analytically we show that the qualitative features of the phase diagram, including the new phase recently found by Martins *et al.*, can be predicted by going one-step beyond the mean field approximation. Numerically we find unusual finite size effects which make the extrapolations to the thermodynamic limit difficult.

In recent years cellular automata (CA) have attracted interest from many researchers because of their practicality in simulating differential equations and because of their fascinating intrinsic dynamics [1, 2]. Normally these automata obey deterministic updating rules. However, there exist a class of automata known as probabilistic cellular automata (PCA) in which the updating rules are chosen randomly from a given probability distribution at each time step. One of the best known of these PCA is the so-called Domany-Kinzel PCA [3].

The Domany-Kinzel PCA consists of a 1-dimensional chain of N spins, n_i, taking on the values $\{0, 1\}$ (empty, occupied). All of the spins are updated simultaneously at discrete time steps and the state of each spin at time $t + 1$ depends only upon the state of the two nearest neighbor spins at time t according to the following rule:

$$P(n_i \mid n_{i+1}, n_{i-1}) =$$
$$\frac{1}{2}\left\{1 - (2n_i - 1)\left[1 - 2p_1(n_{i+1} + n_{i-1}) + 2(2p_1 - p_2)n_{i+1}\,n_{i-1}\right]\right\} \qquad (1)$$

where $P(n_i \mid n_{i+1}n_{i-1})$ is the conditional probability that site i takes on the value n_i given that its neighbors have the values n_{i+1} and n_{i-1} at the previous time step. p_1 (p_2) is the probability that site i is occupied if exactly one (both) of its neighbors is (are) occupied. If neither neighbor is occupied, the site i will also become empty, therefore the state with all sites empty is the absorbing state of the PCA.

The original work of Domany and Kinzel demonstrated the existence of two phases, a frozen phase and an active phase [3]. In the frozen phase all sites become empty in the long time limit
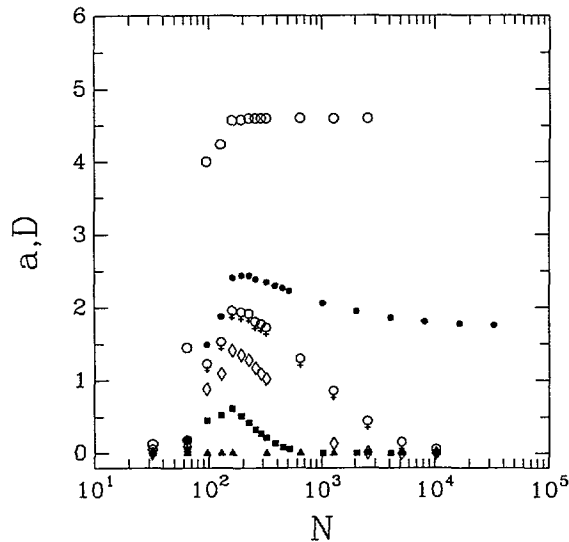
Fig.1. — Plot of the order parameters a and D as a function of system size at $p_2 = 0$. The curves from bottom to top correspond to $p_1 = 0.5$, 0.75, 0.781, 0.797, 0.813, 0.938.

(i.e., all initial states lead to the absorbing state), while in the active phase, for all times, a macroscopic fraction of the sites will be occupied. Their work was based upon the technique of solving the transfer matrix equations and then extrapolating the results to the limit $N \rightarrow \infty$. Since this technique was only applied to small systems, $N \sim 10$, the extrapolations were not very reliable and in fact their paper is ambiguous with regard to the line $p_2 = 0$.

Recently, Martins *et al.* [4] reinvestigated this model via numerical simulations on systems up to $N = 3200$, and found that the active phase can be split into two phases, one chaotic and one non-chaotic. The existence of the chaotic phase was found by taking two initial configurations which differ in a finite fraction of spins and simulating them subject to the same noise. In active (and of course the frozen) phase, the two initial configurations will become identical after some time, however, in the chaotic phase the two initial configurations will *not* become identical.

The purpose of the present work is to develop a better understanding of this new phase via larger numerical simulations and with the help of a novel analytical treatment. As a first step we have performed simulations along the line $p_2 = 0$, in order to resolve the ambiguity mentioned in the Domany-Kinzel analysis. Furthermore, as will be explained later, the activity, a, and the normalized Hamming distance, D, (concentration of spins which differ in the two configurations n and n') along this line are equivalent, thus simplifying the analysis. Initially, n and n' differ only at one site.

Figure 1 shows the value of a, D as function of system size, N, for periodic boundary conditions. In contrast to the work of Martins *et al.*, we simulate the finite PCA only up to $N/2$ time steps, hence, the data shown correspond to that of an infinite system simulated up to $t = N/2$ time steps. It can be seen that the order parameters are not monotonic functions of the system size, rather there is a maximum for a and D in the range $100 < N < 200$. For this reason, investigations on smaller systems may have led to wrong conclusions regarding the location of the transition point. Figure 2 shows our extrapolated values of the order parameter as a function of p_1. The data indicates a second order phase transition, however, at the present
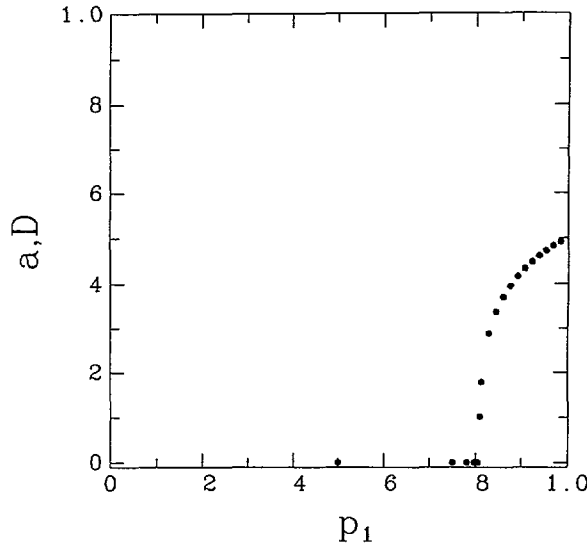
Fig.2. — Plot of the order parameters a and D as a function of p_1 at $p_2 = 0$.

level of accuracy, it is not possible to confirm the previous exponent of $\beta = 0.25$ [3], since the data is also consistent with $\beta = 0.0$.

As a next step we made approximative calculations for the order parameters to determine the qualitative features of the phase diagram, especially for the new chaotic phase. The starting point for an analytical analysis is the complete (conditional) transition probability, $W(n_i, n_i' \mid n_{i+1}, n_{i-1}, n_{i+1}', n_{i-1}')$, for the two configurations to be compared, n and n', subject to the same noise:

$$
\begin{aligned}
W(n_i, n_i' &\mid n_{i+1}, n_{i-1}, n_{i+1}', n_{i-1}') = \\
&\frac{1}{4}\Big\{ 1 - (2n_i - 1)\big[1 - 2p_1(n_{i+1} + n_{i-1}) + 2(2p_1 - p_2)n_{i+1}\,n_{i-1}\big] \\
&- (2n_i' - 1)\big[1 - 2p_1(n_{i+1}' + n_{i-1}') + 2(2p_1 - p_2)n_{i+1}'\,n_{i-1}'\big] \\
&- (2\psi_i - 1)\big[1 - 2p_1(\psi_{i+1} + \psi_{i-1}) - p_2(n_{i+1} - n_{i+1}')(n_{i-1} - n_{i-1}') \\
&+ (p_2 + 4p_1(1 - p_2))\psi_{i+1}\psi_{i-1} + p_2(2p_1 - 1)((n_{i+1} + n_{i+1}')\psi_{i-1} + (n_{i-1} + n_{i-1}')\psi_{i+1})\big]\Big\}
\end{aligned}
\tag{2}
$$

where $\psi = n + n' - 2n\,n'$. This equation can be checked simply by inserting all $2^6 = 64$ possible configurations of the 6 spins. Note also, that setting $p_2 = 0$, yields a transition probability which is symmetric in n, n' and ψ. Thus the activity and the damage order parameters are the same.

The time evolution of the probability $P_t(\{n, n'\})$ to find the system in state $\{n, n'\}$ at time t is then governed by the master equation

$$
P_{t+1}(\{n, n'\}) = \sum_{\{m, m'\}} \prod_{k=1}^{N} W(n_k, n_k' \mid m_{k+1}, m_{k-1}, m_{k+1}', m_{k-1}') P_t(\{m, m'\})
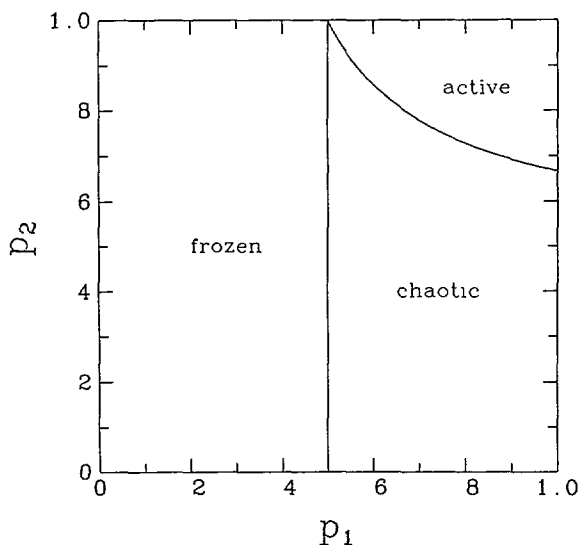\tag{3}
$$

Fig.3. — Phase diagram in the simple mean-field limit.

Expectation values, $\langle f(n,n') \rangle_t$, at time t are simply defined by:

$$\langle f(n,n') \rangle_t = \sum_{n,n'} f(n,n') \, P_t(\{n,n'\}) \tag{4}$$

where the summation extends over all configurations of the n_i and n_i' with $i = 1,..,N$.

In the simplest approximation all correlation functions are decoupled (i.e. $\langle n_i n_j \rangle = \langle n_i \rangle \langle n_j \rangle$) and one gets the iteration equations for the expectation values $a_t = \langle n \rangle_t$ and $D_t = \langle \psi \rangle_t$.

$$a_{t+1} = 2p_1 a_t - (2p_1 - p_2)a_t^2 \tag{5}$$

$$D_{t+1} = 2p_1 D_t - \frac{1}{2}(p_2 - 4p_1(1 - p_2))D_t^2 + 2p_2(1 - 2p_1)a_t \, D_t. \tag{6}$$

The result for the critical lines is shown in figure 3. As can be seen the chaotic region extends up to the line $p_2 = 1$ yielding a triple point on that line in contradiction to the work of Martin *et al.* and our own computer simulations.

Therefore we went one step beyond the mean-field approximation and applied the so-called two tree approximation, known from the studies of directed polymers [5]. In this scheme the correlation functions are decoupled only after every second time step. We do not show the iteration equations themselves because they are to involved, especially for ψ_t. This improved approximation gives rise to a different behavior as shown in figure 4. The tricritical point now is located on the $p_2 = 0$ line as found in the simulations. It should be pointed out that this approximation gives reentrant behavior both in p_1 *and* p_2 direction, a phenomenon which is completely absent in the numerical simulations where both critical lines are monotonic functions of p_1 and p_2. Nevertheless, these approximations support the existence of the chaotic phase. Numerically we have confirmed that the phase diagram is qualitatively the same as found in references [3, 4].

As a summary we shown that the numerical simulations need to be done on larger systems in order to overcome the finite size effects. And we found that the simple mean-field approach
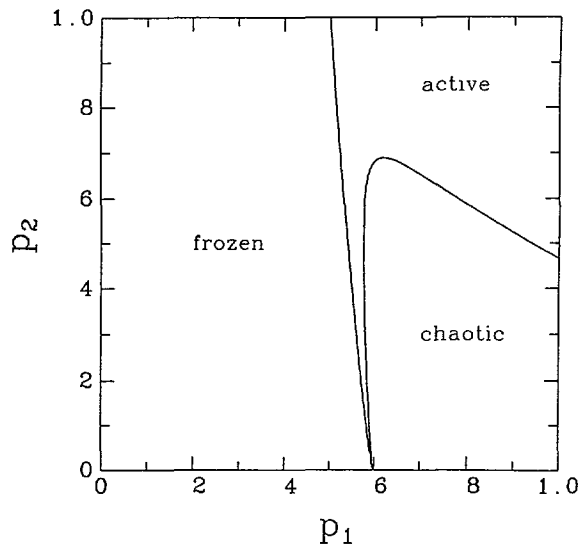
Fig.4. — Phase diagram in the 2-tree approximation.

does not give the right qualitative behavior of the phase diagram. However, it is possible to systematically improve the approximation which is closer to the numerical simulations.

Acknowledgements.

References

[1] Wolfram S., Theory and Applications of Cellular Automata (Singapore: World Scientific, 1986).

[2] Stauffer D., *Phys. Scr.* **T35** (1991) 66.

[3] Domany E. and Kinzel W., *Phys. Rev. Lett.* **53** (1984) 311; Kinzel W., *Z. Phys.* B 58 (1985) 229.

[4] Martins M.L., Verona de Resende H.F., Tsallis C. and de Magalhães A.C.N., *Phys. Rev. Lett.* **66** (1991) 2045; Tsallis C., to appear in "Condensed Matter Theories", Vol. 8, L. B lum Ed. (New York: Plenum, 1992).

[5] Cook J. and Derrida B., *J. Phys. A: Math Gen.* **23** (1990) 1523.