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Metric properties of the Bethe lattice and the Husimi cactus

J. A. de Miranda-Neto and Fernando Moraes

Departamento de Física, UFPE, 50739 Recife, PE, Brazil

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Abstract. — The Bethe lattice and the Husimi cactus, traditionally viewed as graphs embedded in infinite-dimensional spaces, have been used to model a variety of problems. However, only their topological and connectivity properties are used in such models where the idea of metric distance between vertices and bond angles is meaningless. By following the indication of Mosseri and Sadoc that such lattices can be embedded in two-dimensional hyperbolic spaces, we obtain metric properties for those structures and discuss further applications.

1. Introduction.

In the past few decades, the Bethe lattice [1] has had great importance in the process of understanding a variety of phenomena, such as magnetic phase transitions [2], vibrational [3] and electronic [4] properties of solids, percolation [5, 6], gelation [6], cluster growth [6], etc. It provides a model substrate where many problems can be solved exactly. However, only its topological and connectivity properties have been used since it is not a lattice in the conventional sense. A Bethe lattice is usually viewed [2] as a ramified tree embedded in an infinite-dimensional Euclidean space, with a constant vertex connectivity, a total absence of closed rings and having all vertices equivalent. Seen this way, it is not possible to assign coordinates to the vertices, or to measure distances and bond angles on the Bethe lattice. This aspect was emphasized by Thorpe in his lecture at the 1982 NATO summer school on excitations in disordered systems [7] that we quote here :

« Up until now all the problems that we have considered have involved only the topology or connectivity of the pseudo-lattice. It has only been important what is connected to what, and not what angles were. Also we have not attempted to define *distance* in the Bethe lattice. In fact it is impossible to define distance in any meaningful way so that although we have been able to calculate densities of states, it is *not* possible to calculate the neutron scattering as this involves k that is the conjugate variable to a distance ».

Contrary to this view, we show that it is *possible* to define distances in a Bethe lattice and we calculate its radial distribution function, a function that is closely related to the neutron scattering. This is made possible because hierarchical lattices such as the Bethe lattice and the Husimi cactus [8] can be embedded in two-dimensional metric spaces.

We would like to emphasize the distinction between topological (or chemical) and geodesic

distances. The first one denotes the generation or shell that a given vertex belongs to [7]. Notice that the topological distance, since it is a counting of steps or bonds *on the lattice*, is independent of the geometric structure of the space the lattice is embedded in. When the hierarchical lattice is embedded in a space endowed with well defined metric properties, the geodesic distance appears naturally as a characteristic of that space. The geodesic distance between any two given vertices is the length of the geodesic that joins such vertices. Being a geometrical quantity, it takes into account angles and bond lengths (that is not done by the topological distance).

In part 2 of this work we describe the embedding of the hierarchical lattices in two-dimensional metric spaces. This approach allows us to study further characteristics of those lattices which would not make sense in the usual topological description. The calculation of distances and radial distribution functions in such structures is presented in part 3. Finally, in part 4 we discuss our results and comment on the applications of hierarchical lattices endowed with metric properties.

2. Hierarchical lattices and hyperbolic geometry.

Mosseri and Sadoc [9] pointed out that one needs not an infinite-dimensional Euclidean space to describe the Bethe lattice. A simpler two-dimensional space of constant negative curvature is able to support that structure. This space is the hyperbolic or Lobachevsky plane [10-12] (H2), which cannot be embedded in ordinary three-dimensional Euclidean space and hence can only be visualized through suitable projective models. One such a model is a conformal mapping of the whole hyperbolic plane onto a disc, the Poincaré disc, which will be used throughout this work. In this projective model, H2 is represented by a disc whose boundary is denominated the Absolute. This limiting circle represents the points of H2 at infinity and the orthogonal arcs to the Absolute, the geodesics. While angles are preserved in this representation, distances are more and more deformed as one goes from the center of the disc towards the boundary.

The Bethe lattice appears naturally in H2 as a limit case of its tiling by regular polygons. These tilings, or tessellations, are denoted by the Schläfli [13] symbol $\{p, q\}$, meaning the structure obtained by placing q polygons of p edges around each vertex. It is easily shown [14] that if $A = (p - 2)(q - 2)$ one has the following cases :

- a) $A = 4$, euclidean tessellations ;
- b) $A < 4$, spherical tessellations ;
- c) $A > 4$, hyperbolic tessellations.

In the hyperbolic case it is possible to construct honeycombs even in the case either p or q is ∞ . In particular the honeycombs $\{\infty, q\}$ are tilings of H2 by regular apeirogons (polygons of an infinite number of edges), q of them around each vertex defining the bond angles as $2\pi/q$. These are the hyperbolic representations of q -coordinated Bethe lattices. In figure 1 we show the sequence of threefold coordinated lattices whose limit, $\{\infty, 3\}$, is a Bethe lattice. The quasi-regular structure obtained by joining the mid edges of $\{\infty, 3\}$, the Husimi cactus, denoted by $\left\{ \begin{smallmatrix} 3 \\ \infty \end{smallmatrix} \right\}$, is shown in figure 2. Since both structures come from limit cases of regular/quasi-regular tilings, the distances between nearest neighbours and the angles between adjacent bonds are fixed. Viewed as regular or quasi-regular tessellations of H2, those structures can have coordinates assigned to their vertices and distances can thus be measured with the help of non-Euclidean hyperbolic geometry.

We use barycentric coordinates [13] on the projective plane (i.e., the Euclidean plane with the addition of a line at infinity). This coordinate system requires the use of a triangle of

Fig. 1. — Threefold coordinated lattices $\{p, 3\}$ defined on the sphere ($p < 6$), on the plane ($p = 6$) and on the hyperbolic plane ($p > 6$) as represented by the Poincaré disc and the Bethe lattice. In what concerns the hyperbolic lattices the bond lengths are all the same size, they seem deformed due to the projection.

reference on whose vertices we put « masses » t_0 , t_1 and t_2 . Fixing the vertices of that triangle, any point of the projective plane can be described by the set (t_0, t_1, t_2) .

The Absolute is represented on the projective plane by a conic Ω (i.e., a quadratic equation of the coordinates, $\Omega = \sum c_{\mu\nu} x_\mu x_\nu$). For simplicity, we take

$$\Omega = x_0 x_1 + x_1 x_2 + x_0 x_2 = 0. \quad (1)$$

Fig. 2. — The Husimi cactus $\left\{ \begin{smallmatrix} 3 \\ \infty \end{smallmatrix} \right\}$ in the Poincaré disc.

For an equilateral triangle of reference, assigning the coordinates $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ for its vertices, the above equation describes a circle, the boundary of the Poincaré disc. Points satisfying $x_0 x_1 + x_1 x_2 + x_0 x_2 > 0$ are inside the disc and hence represent points of H_2 .

3. Metric properties of the hierarchical lattices.

The coordinates of $\{\infty, 3\}$ are given [15] by the odd solutions of the Diophantine equation

$$x_0 x_1 + x_1 x_2 + x_0 x_2 = 3. \quad (2)$$

Starting with the solution $(1, 1, 1)$, the vertex at the centre of the disc, the operations $(x_0, x_1, x_2) \rightarrow (x_2, x_0, x_1)$ and $(x_0, x_1, x_2) \rightarrow (-x_1, x_0 + 2x_1, x_2 + 2x_1)$ generate all possible solutions to (2). The first few points generated by this procedure plus the triangle of reference are shown in figure 3.

The coordinates of $\left\{ \begin{smallmatrix} 3 \\ \infty \end{smallmatrix} \right\}$ are given by the integral solutions of

$$x_0 x_1 + x_1 x_2 + x_0 x_2 = 1. \quad (3)$$

Similarly, the solutions to (3) are generated by the operations $(x_0, x_1, x_2) \rightarrow (x_2, x_0, x_1)$ and $(x_0, x_1, x_2) \rightarrow (x_1 + 2x_0, -x_0, x_2 + 2x_0)$ beginning with the vertex $(1, 0, 1)$.

Having found the coordinates of the hierarchical lattices, we show now how to obtain analytical expressions for geodesic distances between pairs of vertices on those structures.

The geodesic distance between two points x and y in H_2 is [10]

$$d_{xy} = \kappa^{-1} \operatorname{arc} \cosh \sqrt{\{xy, YX\}}. \quad (4)$$

Where the real constant κ , related to the Gaussian curvature K by $\kappa = \sqrt{-K}$, sets the unit of length in H_2 and $\{xy, YX\}$ is the cross-ratio between the points x and y and their polars X and Y . The polar X of a point x is the line obtained from the transformation $X_\mu = \sum c_{\mu\nu} x_\nu$, $\mu = 0, 1, 2$. The transformation matrix $c_{\mu\nu}$ obtained directly from the equation of the Absolute (1), is

$$c_{\mu\nu} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad (5)$$

Fig. 3. — The Poincaré disc including the triangle of reference and the coordinates of the first few vertices of the Bethe lattice.

The above mentioned cross-ratio is

$$\{xy, YX\} = \frac{\{xY\} \{yX\}}{\{xX\} \{yY\}}, \quad (6)$$

where $\{xY\} = x_0 Y_0 + x_1 Y_1 + x_2 Y_2$, and so on.

We found expressions in very compact form for distances between given vertices in both lattices, which are given below :

$$d = \kappa^{-1} \operatorname{arc} \cosh \left(\frac{x_0 + x_1 + x_2}{3} \right), \quad (7)$$

for the geodesic distance between a generic vertex (x_0, x_1, x_2) and the lattice point $(1, 1, 1)$ in the Bethe lattice $\{\infty, 3\}$;

$$d = \kappa^{-1} \operatorname{arc} \cosh \left(\frac{x_0 + 2x_1 + x_2}{2} \right), \quad (8)$$

for the geodesic distance between a generic vertex (x_0, x_1, x_2) and the lattice point $(1, 0, 1)$ in the Husimi cactus $\left\{ \begin{smallmatrix} 3 \\ \infty \end{smallmatrix} \right\}$. Since in both lattices all the vertices are equivalent, instead of measuring distances between two generic vertices it is more convenient to choose a given vertex as the origin and measure the distances to it. Thus the choice $(1, 1, 1)$ for the Bethe lattice and $(1, 0, 1)$ for the Husimi cactus.

This development permits the calculus of the radial distribution functions (RDF) [16] for the Bethe lattice and the Husimi cactus directly by measuring geodesic distances between pairs of vertices in H2, with the help of equations (7) and (8). The RDF is defined [17] in analogy with the flat space one as the number of vertices with geodesic distances between r

Fig. 4. — Radial distribution function for the Bethe lattice. The radius r is in units of κ^{-1} $g(r)$ in arbitrary units.

Fig. 5. — Radial distribution function for the Husimi cactus. The radius r is in units of κ^{-1} $g(r)$ in arbitrary units.

and $r + dr$ from a given one divided by the area of the corresponding annulus : $2 \pi [\sinh (\kappa r) / \kappa] dr$. In figures 4 and 5 we present the RDF peaks corresponding to the first five shells of the Bethe lattice and the Husimi cactus, respectively. The unit length was chosen such that $\kappa = 1$.

4. Concluding remarks.

The Bethe lattice and the Husimi cactus as commonly used in statistical mechanics are graphs embedded in an infinite-dimensional euclidean space. By embedding these hierarchical structures in the hyperbolic plane we gave them metric properties while keeping their connectivity and topological characteristics. We would like to emphasize additional aspects that appear under this new view of the hierarchical lattices. In particular, to the topological distance (the only one considered in conventional Bethe lattice and Husimi cactus models) between a vertex taken as the origin and the vertices of a given shell, one adds the geodesic distance, a geometrical measure of length that takes into account the particular orientation of the vertices relative to each other. The inclusion of the geodesic distance makes explicit the splitting of the shells, each one defined by a given topological distance, into subshells at different geodesic distances as clearly seen in tables I and II. Notice also the interpenetration of the subshells, a result of the intensity of the splitting. In the case of the cactus, the splitting is such that the radii of subshells belonging to different shells occasionally coincide. All this is

Table I. — *The first five shells of the Bethe lattice.*

Shell number (topological distance from the origin)	Number of vertices	Geodesic distance from the origin (1, 1, 1) (in units of κ^{-1})
1	3	1.0986
2	6	1.9733
3	6	2.6339
	6	2.9591
4	6	3.1480
	12	3.6714
	6	3.9116
5	6	3.5640
	12	4.2094
	6	4.4227
	12	4.5984
	6	4.7004
	6	4.8777

Table II. — *The first five shells of the Husimi cactus.*

Shell number (topological distance from the origin)	Number of vertices	Geodesic distance from the origin (1, 0, 1) (in units of κ^{-1})
1	4	0.9624
2	4	1.7627
	4	1.9248
3	4	2.3895
	8	2.7036
	4	2.8873
4	4	2.8873
	8	3.2945
	4	3.5255
	4	3.6369
	8	3.6629
	4	3.8497
5	4	3.2945
	8	3.7607
	8	4.1429
	4	4.1894
	8	4.2483
	4	4.4187
	8	4.4658
	8	4.5950
	8	4.6249
	4	4.8121

consequence of the high symmetry of the Bethe lattice and the Husimi cactus and the geometry they are submitted to.

This unusual view of a hierarchical lattice is not in contradiction with the infinite-dimensional representation. This is a consequence of the very nature of the hyperbolic two-dimensional space where the volume grows exponentially with the distance, a property that makes its metric properties effectively infinite-dimensional as pointed out by Callan and Wilczek [18]. The topology of space is not affected by the negative curvature. Thus, the metric properties are effectively infinite-dimensional while the topological properties are low-dimensional.

We would like to stress that, to the already powerful tools that are the hierarchical lattices and cacti, one may add now metric and symmetry properties to model systems defined on them. Seen as real lattices in non-Euclidean space, they may be useful for structural modeling of frustrated systems of same coordination, bond angles and dimensionality along the lines set forward by Kléman and Sadoc [19, 20]. For example, threefold coordinated structures with sp^2 bonding may be described with the help of $\{\infty, 3\}$. A word of caution : for modeling fourfold coordinated structures with sp^3 bonding, the $\{6, 3, 3\}$ tessellation of the tridimensional hyperbolic space described by Kléman and Donnadiou [21, 20] is certainly more suitable than $\{\infty, 4\}$ since in addition to the right coordination it has the correct angles and dimensionality. Notice that $\{6, 3, 3\}$ is [21] the three-dimensional analogue of $\{\infty, 3\}$.

One may also think of studying the dimer and Ising problems on Bethe lattices and Husimi cacti by a method that explores their symmetries, the Pffafian [22] one, as done by Lund *et al.* [23] for a particular hierarchical structure also embedded in H2. As an additional refinement we would suggest the introduction of metric characteristics into the Hamiltonian. On the other hand, problems involving high critical dimension such as spin glass and percolation problems may find in the hyperbolic plane a natural setting where mean field theory may be valid [18]. Finally, let us mention that the view we present of the hierarchical lattices permits the construction of discrete models, and thus the use of Monte Carlo techniques, for field theories defined on H2.

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