Properties of Fermi liquids with a finite range interaction

Philippe Nozières

Classification
Physics Abstracts
05.30 — 71.10

Properties of Fermi liquids with a finite range interaction

Philippe Nozières

Institut Laue-Langevin, BP 156, 38042 Grenoble Cedex 9, France

(Received 23 October 1991, accepted in final form 3 January 1992)

Résumé. — Partant d'une suggestion de Khodel' et Shaginyan (KS), on montre que la description d'un liquide de Fermi en Hartree Fock peut conduire à des résultats très étranges quand la portée de l'interaction est grande. Par exemple, la discontinuité de la distribution des particules au niveau de Fermi est étalée sur une bande de k finie, avec un plateau de l'énergie des quasi-particules. En fait, cet état est une conséquence de l'approximation de Hartree Fock. Il se produit seulement pour une attraction, auquel cas il est masqué par la supraconductivité. De plus, le renforcement des collisions entre quasiparticules rend l'approximation de Hartree Fock inutilisable. Enfin, l'écrantage d'une interaction forte et à longue portée ne permet pas d'atteindre le seuil d'instabilité.

Abstract. — Following a suggestion of Khodel' and Shaginyan (KS), it is shown that a Hartree Fock description of Fermi liquids can lead to very strange results when the interaction has long range. For instance, the sharp drop of particle distribution at the Fermi level can be smeared over a finite k-range, with a flat plateau in the quasiparticle energy. In practice, such an effect appears as an artefact of the Hartree Fock approximation. The KS effect occurs only for an attraction, in which case it is hidden by superconductivity. Moreover, the enhanced quasiparticle collision rate makes the Hartree Fock approximation untenable. Finally, screening of a strong long range interaction is such that the instability threshold cannot be reached.

It is a widely accepted belief that treating a Fermi liquid within Hartree Fock approximation can only result in a standard Fermi Dirac distribution, with a sharp jump of the particle distribution n_k from 1 to 0 at a well defined Fermi surface in reciprocal space. Recently, Khodel and Shaginyan [1] have shown that it is not necessarily so : if the interaction has a *finite range* in real space (i.e. V_k localized in **k**-space), the Fermi surface can « broaden » at $T = 0$, the distribution n_k going smoothly from 1 to 0 in a finite range (k_1, k_2). In this brief note, we explore this puzzling result further. Looking at simple limits, we confirm the validity of the KS result within a *normal state Hartree Fock approximation*. We sketch a phase diagram in terms of the strength and range of the interaction. We briefly discuss properties of such a wild looking state. In particular, we show that the quasiparticle energy is constant between k_1 and k_2.

We then argue that in practice such a normal Hartree Fock state is not tenable for a number of reasons.

(i) Fermi surface broadening only occurs for *attractive* interaction between particles, in which case BCS pairing should occur. The ground state is superconducting, with a « natural » Fermi surface broadening — albeit very different from the usual one.

(ii) If we ignore the superconducting instability, the flat plateau in ε_k enhances quasiparticle collisions drastically : a Hartree Fock approximation is clearly inaccurate.

(iii) The bare interaction is actually reduced by screening. For long range coupling the screened interaction never reaches the KS threshold.

As a result, the exotic KS state should not occur under reasonable conditions : Fermi liquid behaviour is preserved. It remains that the usual Hartree Fock reference state is patently wrong : it is instructive to know what should happen if it were valid.

1. The normal Hartree Fock state.

At this stage, spin is irrelevant : we discard it and we describe an Hartree Fock state in terms of the particle distribution n_k. The energy is

$$E - \mu N = \sum_{\mathbf{k}} (\xi_k - \mu) n_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{kk}'} V_{\mathbf{kk}'} n_{\mathbf{k}} n_{\mathbf{k}'} \tag{1}$$

ξ_k is the kinetic energy, μ the chemical potential. $V_{\mathbf{kk}'}$ is characterized by its range $|\mathbf{k}' - \mathbf{k}| \simeq \sigma$, and by its strength, conveniently measured as

$$\sum_{\mathbf{k}'} V_{\mathbf{kk}'} = U \tag{2}$$

(U is the interaction energy at zero particle distance). At first sight, $U > 0$ means a repulsion : in practice, it is not that obvious : see section 2. For a contact interaction in real space, $V_{\mathbf{kk}'} = V = \text{Cst.}$ yielding the usual Hubbard model. If $V(r)$ has a range ρ, then $\sigma \sim 1/\rho$. The renormalized quasi-particle energy is

$$\varepsilon_k = \frac{\partial E}{\delta n_{\mathbf{k}}} = \xi_k - \mu + \sum_{\mathbf{k}'} V_{\mathbf{kk}'} n_{\mathbf{k}'} . \tag{3}$$

The equilibrium distribution at temperature T minimizes $E - TS$, where S is the standard entropy

$$S = - \sum_{\mathbf{k}} [n_{\mathbf{k}} \log n_{\mathbf{k}} + (1 - n_{\mathbf{k}}) \log (1 - n_{\mathbf{k}})] \tag{4}$$

((4) measures the configurational freedom in building an intermediate $n_{\mathbf{k}}$). $n_{\mathbf{k}}$ is just the usual Fermi distribution $f(\varepsilon_k)$.

In practice, σ should be a fraction of k_{F}. Nevertheless, we first look at extreme cases in order to clarify the physics. If σ is infinite (i.e. a contact interaction) the interaction energy $V_{\mathbf{kk}'} n_{\mathbf{k}'}$ is just a constant VN which may be absorbed in the chemical potential μ : the behaviour is that of a free Fermi gas, as usual. In the opposite case $\sigma \to 0$ (i.e. a long range interaction), (1) and (3) reduce to

$$\begin{cases} E - \mu N = \sum_{\mathbf{k}} \left\{ (\xi_k - \mu) n_k + \frac{1}{2} U n_k^2 \right\} \\ \varepsilon_k = \xi_k - \mu + U n_k . \end{cases} \tag{5}$$

Each **k** value behaves on its own.

Such a limit $\sigma \to 0$ calls for some comment. If σ were identically zero, i.e. $V_{\mathbf{kk}'} = U\delta_{\mathbf{kk}'}$, then the problem would be trivial. The interaction energy would be a constant $N(N-1)U/2$ (non extensive), and the Schrödinger equation would be that of free fermions. What we have in mind here is a limit where $\sigma \ll k_F$, yet much larger than the *level spacing* $\sim 1/N$ in k space. The summation in equation (2) should cover a *large* number of k' states in such a way as to introduce the *average* occupancy n_k. Yet σ should be small enough that n_k varies little on that scale. In this way, we retain a thermodynamically extensive problem, with a *finite* range of interactions in real space.

The ground state depends critically on the sign of U, as shown in figure 1. When U is < 0, the energy is minimal for either $n_k = 0$ or $n_k = 1$: a sharp « first order » transition occurs

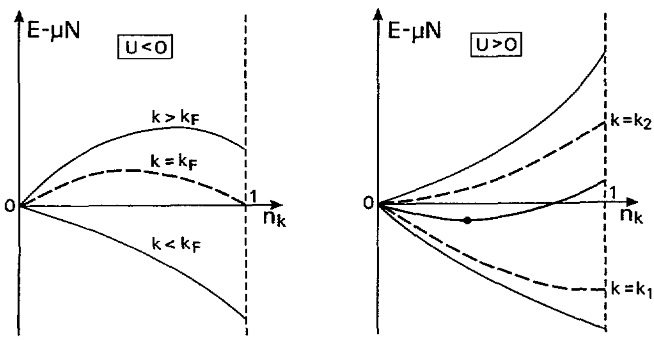

Fig. 1. — The ground state energy as a function of n_k for an interaction with $\sigma = 0$.

at some $k = k_F$. We recover the usual Fermi distribution. In the opposite case $U > 0$, the minimal energy corresponds to an intermediate n_k in a finite range of k, $k_1 < k < k_2$; the Fermi discontinuity is *smeared* at $T = 0$, a conclusion that looks crazy at first sight. The result is simply

$$n_k = \frac{\xi_k - \mu}{U}, \quad \varepsilon_k = 0 \tag{6}$$

(k_1 and k_2 correspond to those values of ξ_k for which n_k is respectively 1 and 0). Equivalently, one may obtain (6) through a graphical solution of the coupled equations for n_k and ε_k.

$$\begin{cases} \varepsilon_k = \xi_k - \mu + U n_k \\ n_k = f(\varepsilon_k) = \theta(-\varepsilon_k) . \end{cases}$$

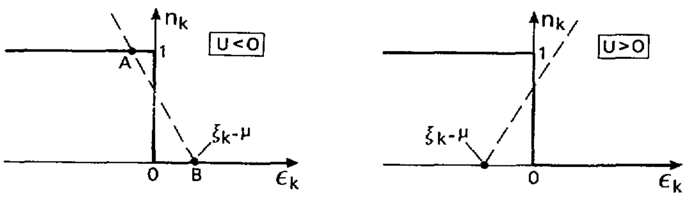

Fig. 2. — A graphical construction of the ground state distribution n_k for an interaction with $\sigma = 0$.

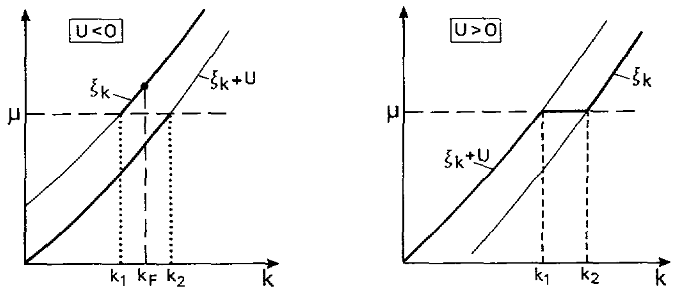

Fig. 3. — The quasiparticle energy ε_k as a function of k for either a filled state ($n_k = 1$) or an empty state ($n_k = 0$).

The solution is shown in figure 2. When U is > 0, the plateau in ε_k is obvious. When U is < 0, there are two stable solutions in the intermediate range, with a first order jump from one to the other at $k = k_{\mathrm{F}}$. Still another interpretation is shown in figure 3, which shows the quasiparticle energy, $\mu + \varepsilon_k$, for either $n_k = 0$ or $n_k = 1$. The heavy lines correspond to those states which are consistent at $T = 0$. When $U < 0$, there are two solutions between k_1 and k_2, one of which minimizes the energy. When $U > 0$, there are *no* solutions between k_1 and k_2 . we are forced to look for an intermediate n_k.

The graphical construction of figure 2 is easily extended to finite temperatures, as shown in figure 4. The results are even more appalling.

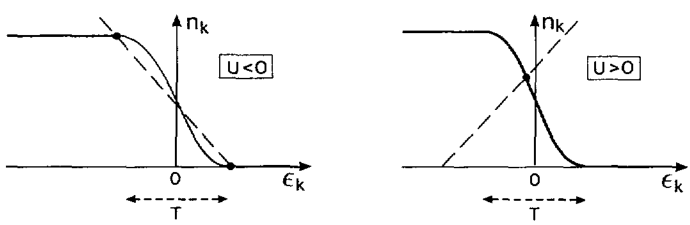

Fig. 4. — Graphical construction of $n_k(T)$ at finite temperatures.

— When $U < 0$, a finite discontinuity remains in n_k at low enough temperatures, up to a critical T^* at which n_k recovers the usual smooth behaviour. This is a mystery.

— When $U > 0$, the plateau in ε_k broadens over an energy range $\sim T$ (see Fig. 5). Everything is smooth — but, still, a finite number of states (roughly between k_1 and k_2) are packed in an energy range $\sim T$, leading to a dramatically high density of states.

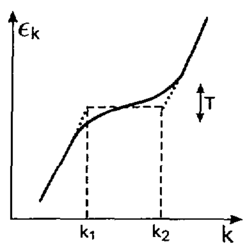

Fig. 5. — The quasiparticle energy ε_k at finite temperatures.

In both cases, the usual Fermi liquid picture breaks down. Even if the Hartree Fock approximation is invalid (which it is !), the physical behaviour should depart drastically from usual models.

Of course, the limit $\sigma = 0$ is unphysical, and we should look for the effect of a finite σ, which will « blur » momenta on that scale. The effect will be small as long as

$$\sigma < \delta = k_2 - k_1 = \frac{U}{v_F} \tag{7}$$

where v_F is the bare Fermi velocity. In such a case, and for $U > 0$, momentum blurring is not enough to bridge the gap in figure 3 : n_k is forced to sit *between* 0 and 1 in the range (k_1, k_2), except may be near the edges. Since $0 < n_k < 1$ implies $\varepsilon_k \equiv 0$ (energy must be stationary), a *rigorously flat plateau* in ε_k remains, despite momentum blurring. Such a flat plateau necessarily has *sharp edges,* $\bar{k}_1$ and $\bar{k}_2$, beyond which n_k is *exactly* 0 or 1. These conclusions are completely independent of the shape of $V_{\mathbf{kk}'}$. they are forced by energy minimization.

Mathematically, one must solve the integral equation

$$\varepsilon_k = \xi_k - \mu + \sum_{\mathbf{k}'} V_{\mathbf{kk}'} n_{\mathbf{k}'}$$

with the constraints that

$$\begin{cases} \varepsilon_k = 0 & \text{if} \quad k_1 < k < k_2 \\ n_k = 1 & \text{if} \quad k < k_1 \\ n_k = 0 & \text{if} \quad k > k_2 \,. \end{cases} \tag{8}$$

The constraints (8) also determine k_1 and k_2, which must be obtained self consistently. It so happens that an exact solution may be found in a specific case which confirms the above general discussion. Let us average $V_{\mathbf{kk}'}$ over the directions of $\mathbf{k}$ and $\mathbf{k}'$, thereby defining an « s-wave » matrix element $\bar{V}(k, k')$ that depends on the moduli k, k'. Then

$$U = \rho v_F \int \bar{V}(k, k')\, \mathrm{d}k' \tag{9}$$

where ρ is the bare density of states at Fermi level (we assume for simplicity that ρ and the quasiparticle velocity $v_k = \partial \xi_k / \partial k$ are constant in the relevant range near Fermi level). We then try a shape

$$\rho v_F \bar{V}(k, k') = \frac{U}{2\sigma} \exp\left[-\frac{|k - k'|}{\sigma}\right] \tag{10}$$

(i.e. the Fourier transform of a Lorentzian). (8) is a Wiener Hopf system for which standard techniques are available. The results are shown in figure 6 :

(i) The quasiparticle energy ε_k does have a flat plateau, with an intermediate n_k, in the range

$$\begin{cases} \bar{k}_1 = k_F - \delta \,, \quad \bar{k}_2 = k_F + \delta \\ \delta = \dfrac{U}{2 v_F} - \sigma \end{cases} \tag{11}$$

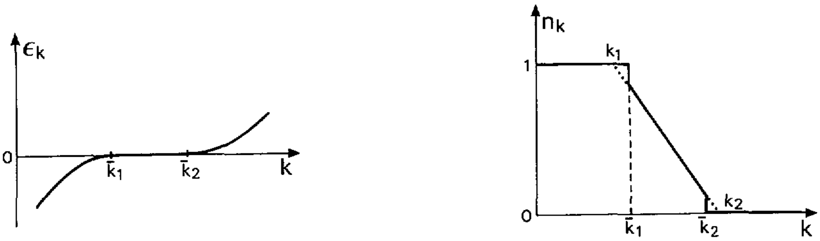

Fig. 6. — The ground state particle distribution n_k and quasiparticle energy ε_k for $\sigma \neq 0$ with the model interaction (9).

(k_F is the Fermi momentum in the absence of interaction). The plateau shrinks for $\sigma \neq 0$ and disappears at a critical $\sigma^* = U/2v_F$. we recover a usual Fermi liquid.

(ii) n_k displays discontinuities at $\bar{k}_1$ and $\bar{k}_2$. it is not clear whether they are artefacts due to the cusp in $\bar{V}$ at $|k| = 0$.

(iii) ε_k is smooth at $\bar{k}_1$ and $\bar{k}_2$, with a zero slope matching to the flat plateau. These results are slightly modified if one allows for a dispersion of v_k (especially the symmetry with respect to k_F) — but the general structure is unchanged. Since we are going to argue that such a state is unrealistic, it seems hardly justified to go into details of that analysis : it is briefly sketched in appendix B.

One may draw a phase diagram in the (U, σ) space, which separates regular Fermi liquids from the KS state. The threshold for the KS instability is reached when the « Fermi liquid » quasiparticle energy

$$\varepsilon_k^{FL} = \xi_k + \sum_{\substack{\mathbf{k}' \\ |k'| < k_F}} V_{\mathbf{kk}'} \, n_{\mathbf{k}'} \tag{12}$$

develops an horizontal inflection point. Detailed values depend on the shape of $V_{\mathbf{kk}'}$ — but the overall picture can be obtained by qualitative arguments. In the limit $\sigma \ll k_F$ considered previously, the interaction term in (12) drops by U in a range $\Delta k \sim \sigma$ — hence a drop in velocity $\sim U/\sigma$. $\mathrm{d}\varepsilon_k^{FL}/\mathrm{d}k$ vanishes when $\sigma = \sigma^* \sim U/v_F$. We recover the previous estimate. In the hypothetical case $\sigma \gg k_F$, we could expand $V_{\mathbf{kk}'}$ as

$$V_{\mathbf{kk}'} = V_0 \left[1 - \frac{(\mathbf{k} - \mathbf{k}')^2}{\sigma^2} + \cdot \quad \right]. \tag{13}$$

The k^2 term in (13) renormalizes the quasiparticle effective mass. The net Fermi velocity vanishes if

$$\frac{NV_0}{\sigma^2} \sim \frac{1}{m} \tag{14}$$

The corresponding diagram is sketched in figure 7 ([1]). In practice, only the region $U/E_F \leq 1$ looks reasonable. The instability condition $U > \sigma v_F$ may then be written in a more

([1]) Actually, the bifurcation may be more complicated if one allows for dispersion in v_k since the inflection point does not necessarily appear at k_F. It may be that, at first, a « hole » is dug inside the filled Fermi sea $k < k_F$, eventually evolving into the smooth drop between k_1 and k_2.

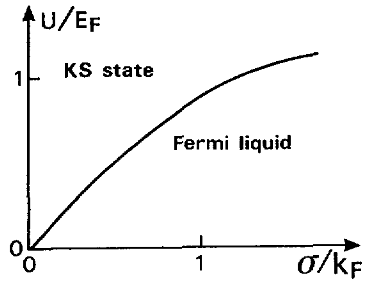

Fig. 7. — The phase diagram for $U > 0$.

transparent fashion using the angle averaged interaction $\bar{V}$. The latter has a maximum $\bar{V}_{\mathrm{m}}$ at $k = 0$ (Fig. 8). From (9) it follows that $U \sim \rho v_{\mathrm{F}} \bar{V}_{\mathrm{m}} \sigma$. The threshold for instability thus corresponds to

$$\rho \bar{V}_{\mathrm{m}} > 1 \tag{15}$$

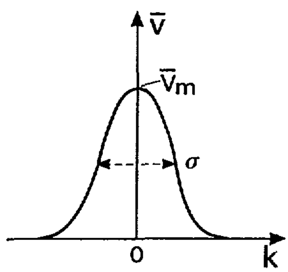

Fig. 8. — The angle averaged particle interaction.

i.e. a *strong coupling regime* (the range σ is hidden in $\bar{V}_{\mathrm{m}}$ at given U).

We will see shortly that such a KS state is unrealistic for a number of reasons. Nevertheless, it is interesting to comment on its physical implications if it happens to exist.

(i) In the KS *normal* state, a finite entropy S_0, given by (4), persists down to $T = 0$. It accounts for the freedom in choosing the filled states among all those between k_1 and k_2.

(ii) The compressibility is unaffected : the KS singularity is attached to the Fermi level, and it moves along as the chemical potential μ changes. As a result, $\mathrm{d}\mu/\mathrm{d}N$ is the same as for a free Fermi gas (a similar situation occurs in the mass renormalization due to electron phonon interactions). We show in appendix A that the whole response function $\chi(q, \omega)$ is unchanged when $\sigma = 0$.

(iii) One would expect the low temperature specific heat to be dramatically affected by the flat plateau in ε_k. Strangely enough, this is not so ! When $\sigma = 0$, the specific heat is the same as that of a free Fermi gas, as shown in appendix A. (Note that the standard wisdom according to which a linear specific heat is the signature of a Fermi liquid is simply wrong !). When $\sigma \neq 0$, this conclusion remains valid as long as $T > \sigma v_{\mathrm{F}}$ (the range in $V_{\mathrm{kk'}}$ is negligible on the thermal scale). What happens when $T < \sigma v_{\mathrm{F}}$ is not clear : the low temperature behaviour can only be worked out numerically. KS claim that the specific heat behaves as $\sqrt{T}$ — a most surprising result. Anyhow, the issue is semantic, as the actual ground state is superconducting, with a finite gap Δ : the specific heat is exponential.

Strangely enough, the KS state does not seem to affect thermodynamical properties.

2. Superconducting pairing.

We now restore spin. The quasiparticle energy should be written in the usual way

$$\varepsilon_{k\sigma} = \xi_k - \mu + \sum_{\mathbf{k}'\sigma'} n_{\mathbf{k}'\sigma'} [V_0 - V_{\mathbf{k}-\mathbf{k}'}\, \delta_{\sigma\sigma'}] \tag{16}$$

V_0 is the Hartree term, which is of no interest to us since it is only a shift in the chemical potential : we discard it. Range effects can only occur in the *Fock exchange term*. But within a Hartree Fock approximation, that term has a *minus* sign. Consequently, $U > 0$ implies an *attractive* interaction, while a repulsion would imply $U < 0$. In the latter case, the strange behaviour at finite T remains, with its mysteries. On the other hand, the KS « normal state » for $U > 0$ is hidden by superconductive pairing, a feature that will necessarily appear as soon as the particles attract. The above discussion must be taken afresh within a BCS approximation : the coherent ground state will be non degenerate, and the zero temperature entropy will disappear — a welcome improvement.

Specifically, we write the ground state as

$$|\psi_0\rangle = \prod_{\mathbf{k}} [u_k + v_k\, c_{k\uparrow}^* \, c_{-k\downarrow}^*]|\text{vac}\rangle \,.$$

Since the phase is locked, we take u_k and v_k as real. We define the averages

$$\begin{aligned} n_k &= \langle c_{k\sigma}^* \, c_{k\sigma} \rangle = v_k^2 \\ x_k &= \langle c_{-k\downarrow} \, c_{k\uparrow} \rangle = u_k\, v_k = \sqrt{n_k(1-n_k)}\,. \end{aligned}$$

The ground state energy is then

$$E - \mu N = \sum_{\mathbf{k}} 2(\xi_k - \mu)\, n_k - \sum_{\mathbf{kk}'} V_{\mathbf{kk}'} [n_{\mathbf{k}}\, n_{\mathbf{k}'} - x_{\mathbf{k}}\, x_{\mathbf{k}'}] \,. \tag{17}$$

Note the change of sign as compared to (1). In (17), we recognize the usual Fock and Bogoliubov terms. It is clear that a finite x_k will lower the energy if $V_{\mathbf{kk}'} < 0$ (i.e. for an attraction).

Minimization with respect to n_k yields the usual BCS formulation

$$\begin{cases} \varepsilon_k = \xi_k - \mu + \sum_{\mathbf{k}'} |V_{\mathbf{kk}'}|\, n_{\mathbf{k}'} \\ \Delta_k = \sum_{\mathbf{k}'} |V_{\mathbf{kk}'}|\, x_{\mathbf{k}'} \end{cases} \tag{18}$$

$$\begin{cases} n_k = \dfrac{1}{2}\left[1 - \dfrac{\varepsilon_k}{E_k}\right], \quad x_k = \dfrac{\Delta_k}{2\,E_k} \\ E_k^2 = x_k^2 + \Delta_k^2 \,. \end{cases} \tag{19}$$

These equations are valid for any $V_{\mathbf{kk}'}$. For a *short range* interaction (in real space), $V_{\mathbf{kk}'}$ is nearly constant, equal to $-V$ up to a cut off $\varepsilon_k \sim \sigma v_F \gg \Delta$: we then recover the usual

BCS situation. The gap Δ is practically constant near the Fermi level, and the quasiparticle energy E_k has the usual hyperbolic shape with a minimum at k_F. Δ and the critical temperature T_c have the usual exponential behaviour in terms of ρV.

The solution is qualitatively different for *long range* interactions. Again, we consider first the extreme case $\sigma = 0$. Then the energy (17) reduces to

$$E - \mu N = \sum_{\mathbf{k}} \{2(\xi_k - \mu) n_k + U n_k [2 n_k - 1]\} \tag{20}$$

where $U = -\sum_{\mathbf{k}'} V_{\mathbf{kk}'} > 0$. (20) is similar to (5), with different coefficients due to the pairing terms. Minimization with respect to n_k is straigthforward. In a finite range k_1, k_2,

$$\left.\begin{cases} n_k = \dfrac{2[\mu - \xi_k] + U}{4U} \\ \varepsilon_k = \dfrac{\xi_k - \mu}{2} + \dfrac{U}{4} \\ \Delta_k = \left[\dfrac{U}{2} + \varepsilon_k\right]\left[\dfrac{U}{2} - \varepsilon_k\right] \end{cases}\right\} \Rightarrow E_k = \frac{U}{2} \tag{21}$$

k_1 corresponds to $n_k = 1$, k_2 to $n_k = 0$, according to

$$\xi_{k_1} = \mu - \frac{3U}{2}, \quad \xi_{k_2} = \mu + \frac{U}{2}. \tag{22}$$

At both points, $\Delta_k = 0$. Outside that range, Δ_k is identically zero. These results are illustrated in figure 9.

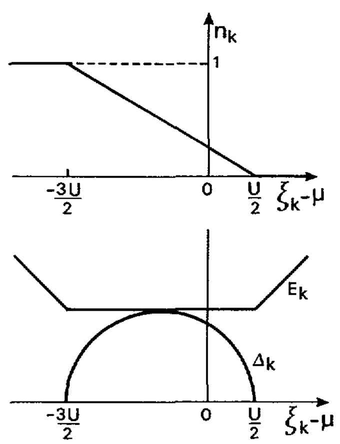

Fig. 9. — The particle distribution and quasiparticle energy in the superconducting ground state for $\sigma = 0$.

n_k drops smoothly from 1 to 0 in a finite range (k_1, k_2). Such a behaviour is familiar in superconductors : the new feature is the existence of sharp boundaries. The distance between these boundaries,

$$\delta = k_2 - k_1 = \frac{2U}{v_F} \tag{23}$$

is proportional to the interaction strength, twice as big as in the « normal » KS state. The striking feature is the flat plateau in E_k, which actually is obvious in (18)

$$\Delta_k = U x_k = \frac{U \Delta_k}{2 E_k} \tag{24}$$

The change in the Fock energy ε_k exactly compensates that of Δ_k, and the superconducting quasiparticles are *dispersionless* in a finite range δ. The resulting density of states is considerably enhanced.

The description (21) is easily extended to finite T. For instance, (24) is replaced by

$$\Delta_k = U \frac{\Delta_k}{2 E_k} \operatorname{th} \frac{E_k}{2 T} \tag{25}$$

from which we infer the critical temperature $T_c = U/4$. (Note that $2 \Delta = 4 T_c$ in that limit). Above T_c, we recover the KS state — but the thermal blurring of the plateau is very large. Using (A.5), we find that the density of states is enhanced by a factor

$$1 + \frac{U n_k (1 - n_k)}{T} \tag{26}$$

which is at most a factor 2 : there is no spectacular effect.

Let us now restore a finite range σ. The structure in k space is blurred on that scale. The edges k_1 and k_2 in the gap are smoothed, and the quasiparticle energy acquires a small dispersion, as shown in figure 10. In order to estimate that dispersion, we write the gap equation (19) as

$$2 E_k x_k = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} x_{\mathbf{k}'} \approx U [x_k - \sigma^2 x_k'' + \cdot \quad] . \tag{27}$$

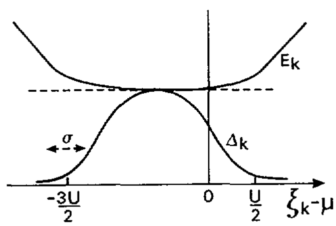

Fig. 10. — The quasiparticle spectrum for a small finite range σ.

According to (27), the dispersion in E_k should be of order $\sigma^2 v_F^2/U$, very small for small σ. Everything is now smooth, but the quasiparticle density of states at $T = 0$ is still large (albeit finite). As T grows this enhancement disappears.

It is instructive to express the zero temperature gap Δ in terms of the parameters σ, $\bar{V}_m$ of figure 8 (which characterize the angle averaged interaction). We have shown that $U \simeq \rho v_F \sigma \bar{V}_m$. One may put the results for small and large σ together by writing

$$\Delta = \sigma v_F \varphi [\rho \bar{V}_m] . \tag{28}$$

For $\rho \bar{V}_m \ll 1$, φ is just the usual BCS exponential. In the opposite limit $\rho \bar{V}_m \gg 1$ (but still with $\sigma \ll p_F$, $U \lessapprox E_F$), φ is linear, yielding $\Delta \simeq U$. The above result thus appears as the strong coupling limit of BCS, in the unusual case where σ is small.

In the end, superconductive pairing destroys the most spectacular effects of the KS state. This is due to the presence of a single interaction kernel $V_{\mathbf{kk}'}$ in (17). The latter feature is specific of the Hartree Fock approximation. If higher order manybody corrections were taken into account, the effective scattering kernels would be different in the particle-particle and particle-hole channels, with an interaction energy

$$\sum_{\mathbf{kk}'} [g^{(1)}_{\mathbf{kk}'} n_{\mathbf{k}} n_{\mathbf{k}'} - g^{(2)}_{\mathbf{kk}'} x_{\mathbf{k}} x_{\mathbf{k}'}] .$$

If $g^{(1)}$ were positive and $g^{(2)}$ negative, a *normal* KS state would hold for small enough σ. In practice, it seems unlikely that renormalization could change signs. More modestly, if the value of g_2 is reduced compared to g_1 the critical temperature is lowered : large KS effects due to the exchange interaction might persist above T_c.

3. The effect of quasiparticle collisions.

We return to the normal state. A mean field Hartree Fock picture necessarily ignores particle collisions, whether in the standard Fermi liquid or in the exotic KS situation. These collisions enter through an imaginary self energy, which in the lowest order is given by the diagram of figure 11. The corresponding lifetime $\tau = 1/\Gamma$ for the quasiparticle $\mathbf{k}$ is given by

$$\Gamma = 2\pi \sum_{\mathbf{qp}} V_q^2 \, \delta[\varepsilon_{\mathbf{k}+\mathbf{q}} + \varepsilon_{\mathbf{p}-\mathbf{q}} - \varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{k}}] \, n_{\mathbf{p}} [1 - n_{\mathbf{p}-\mathbf{q}}][1 - n_{\mathbf{k}+\mathbf{q}}] . \tag{29}$$

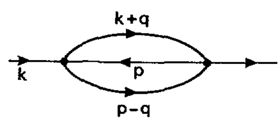

Fig. 11. — The lowest order diagram contributing to quasiparticle lifetime.

In the usual limit $\sigma \gtrsim k_F$, $V_q = V$ is nearly constant. One can break the integrals over $\mathbf{q}$ and $\mathbf{p}$ into energy and angle integrations. The latter provide only numerical factors : qualitatively

$$\sum_{\mathbf{qp}} \sim \int k_F^3 \, d|\mathbf{p}| \, d|\mathbf{p} - \mathbf{q}| \, d|\mathbf{k} + \mathbf{q}| \sim \frac{\rho^3}{N} \int \mathrm{d}\varepsilon_{\mathbf{p}} \, \mathrm{d}\varepsilon_{\mathbf{p}-\mathbf{q}} \, \mathrm{d}\varepsilon_{\mathbf{k}+\mathbf{q}} \tag{30}$$

ρ is the density of states ($\sim k_F^{d-1}/v_F$ where d is the dimension). At low temperatures, we recover the usual result for thermal quasiparticles

$$\Gamma \sim \frac{\rho^3 V^2}{N} T^2 \tag{31}$$

(31) is a standard feature of Fermi liquids.

Consider now the long range case $\sigma \ll k_F$ in the normal « Fermi liquid regime » $U < \sigma v_F$. the Fermi velocity is not changed much by interactions. We expect a crossover when $T \sim \sigma v_F$. Above that crossover, the energy change in the δ-function of (29) is

$\sim \sigma v_F$, much smaller than T. The number of relevant $\mathbf{q}$ values in (29) is $\sim \sigma^d$, while the number of $\mathbf{p}$ values is $\sim \rho T$ (controlled by the occupation factors in (29)). As a result

$$\Gamma \sim \frac{V_0^2 \sigma^d \rho T}{\sigma v_F} \sim \left(\frac{U}{\sigma v_F}\right)^2 \left(\frac{k_F}{\sigma}\right)^{d-1} T \quad (T \gg \sigma v_F) \tag{32}$$

(remember that $U \sim V_0 \sigma^d$). In the opposite limit $T \ll \sigma v_F$, an extra factor $T/\sigma v_F$ enters, due to the occupation factors. We thus obtain

$$\Gamma \sim \frac{V_0^2 \sigma^{d-2} \rho T^2}{v_F^2} \sim \left(\frac{U}{\sigma v_F}\right)^2 \left(\frac{k_F}{\sigma}\right)^d \frac{T^2}{k_F v_F} \quad (T \ll \sigma v_F) . \tag{33}$$

We recover the usual T^2 behaviour, with a coefficient modified by σ. The unexpected result is the dependence $\Gamma \sim T$ when $T > \sigma v_F$, due to a simple counting argument in the golden rule (29). Even if such a range of temperature is not very realistic, that result should be kept in mind.

We now turn to the KS state, occurring when $U > \sigma v_F$. Then the Fermi velocity is drastically renormalized : the bare v_F is replaced by an effective $\tilde{v}_F = \partial \varepsilon_k / \partial k$. The density of states is corrected accordingly ($\tilde{\rho} \sim 1/\tilde{v}_F$). In all our results (31)-(33), v_F should be replaced by $\tilde{v}_F$. the collision rate is considerably enhanced. At $T = 0$, the flat plateau in ε_k implies an infinite Γ. At finite temperatures, the plateau broadens over an energy range $\sim T$, and

$$\tilde{v}_F = v_F \frac{T}{U} \tag{34}$$

Then Γ is finite — but hopelessly large, $\sim 1/T$ in the whole range of temperature. These results make no sense. The origin of the difficulty is apparent if we assume that all momenta in figure 11, $\mathbf{k}$, $\mathbf{p}$, $\mathbf{k}+\mathbf{q}$, $\mathbf{p}-\mathbf{q}$, lie within the plateau of ε_k. Then energy conservation is automatically guaranteed, and the δ-function is infinite. In such a case, the transition probability is meaningless. We should instead consider that the quasiparticle $\mathbf{k}$ is coupled to a *degenerate* configuration (hole $\mathbf{p}$, particles $(\mathbf{k}+\mathbf{q})$ and $(\mathbf{p}-\mathbf{q})$), with a matrix element V_q. These states repel, with an energy splitting $\sim V_q$. Since there is a continuum of excited configurations, a broad spectrum appears. The spectral density $A(k, \varepsilon) = \mathrm{Im}\, G(k, \varepsilon)$ should display a broad structure instead of the usual narrow peak (2).

In practice, the whole KS formulation breaks down. If we allow for particle broadening, the plateau in ε_k does not make sense anymore. What is needed is a *self consistent description,* in which spectral broadening of A is included in the *internal* propagators in figure 11. This is a formidable task which at the moment appears out of reach. It is not clear whether the strange KS state would survive such a more elaborate formulation. But one thing is sure : *the result should have nothing to do with ordinary Fermi liquids.* Quasiparticle broadening has become the central issue, not a marginal complication at low temperatures.

4. The effect of screening.

Until now, we have argued that the KS normal Hartree Fock state was an oversimplified

(2) The situation is somewhat reminiscent of 1-dimensional Fermi liquids when one considers scattering of particles on the *same* side of the Fermi surface. Then momentum conservation implies energy conservation : *all excited configurations available to the particle k are degenerate with it.* That situation was considered long ago by Dzyaloshinski and Larkin [2] : as expected, $A(k, \varepsilon)$ has an elliptic broadened shape, completely different from the usual Lorentzian line shape.

picture. Nevertheless, properties were clearly anomalous when $\sigma < k_F$, $\sigma v_F < U$. It remains to be seen whether such conditions can be achieved in practice. We now show that, even allowing for arbitrary bare parameters σ and U, screening will change the interaction so much that the condition $U < \sigma v_F$ cannot be met under standard conditions.

In calculating the quasiparticle energy (16), we ignored the Hartree term, which was absorbed in μ. We are allowed to do that for the *average* interaction, but not for fluctuations. Indeed, time dependent fluctuations of the Hartree terms are responsible for the RPA screening of V_q, described by the diagrams of figure 12. Due to that screening, V_q should be replaced everywhere by

$$\tilde{V}_q = \frac{V_q}{1 + V_q \chi} \tag{35}$$

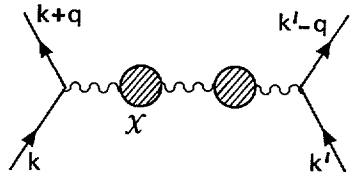

Fig. 12. — Definition of the RPA screened interaction.

where χ is the density-density response function as modified by the Fock term alone. For a repulsion, $V_q > 0$, screening would reduce the interaction. For an attraction, $\tilde{V}_q$ is *enhanced,* leading to a charge instability if $|V_q|\ \chi > 1$ (the system collapses when $|V_0|\ \chi_0 > 1$).

As shown in appendix A, χ is the same as for a free Fermi gas when $\sigma = 0$. Since we are interested in V_q for $q \sim \sigma$, the limit $\sigma = 0$ may not be appropriate. Still, an estimate $\chi \sim \rho$ seems reasonable ([3]). The whole issue is to compare $\rho |V_q|$ with 1.

In the limit $q = 0$, we have (disregarding long range Coulomb effects)

$$\rho V_0 \sim \rho \frac{U}{\sigma^d} \sim \frac{U}{\sigma v_F} \left(\frac{k_F}{\sigma} \right)^{d-1} \tag{36}$$

In order for long range anomalous effects to appear, we would need $\rho V_0 \gg 1$: charge instabilities have appeared long before ([4]).

Admittedly, these arguments are somewhat handwaving. Nevertheless, they leave little hope of observing any KS behaviour. A strong, long range interaction does not survive screening. One could look for anomalous effects when $\sigma \sim k_F$, $U \gg E_F$ — but in that case, anything can happen : a mean field approach is meaningless.

In conclusion, the anomalies predicted by KS do not seem to be realistic, except may be under very pathological conditions. Nevertheless, it is useful to know what would happen if they were to occur — indeed, it is quite a surprise that a plain Hartree Fock solution can be that non trivial.

([3]) The ground state energy involves a *dynamically* screened interaction for $\omega \neq 0$: for a rough estimate, that complication may be ignored.

([4]) Strictly speaking, the KS condition would be met *very close* to the charge instabilities, when $\tilde{V}_0$ is large and the enhancement restricted to small q : such a situation is rather artificial.

Acknowledgements.

The author is grateful to N. Schopohl and G. Volovik for calling his attention to the work of Khodel' and Shaginyan — and for many passionate and fruitful discussions.

Appendix A.

From the entropy (4) we infer the specific heat

$$C_v = T\frac{\partial S}{\partial T} = -T\sum_k \frac{\partial n_k}{\partial T}\log\frac{n_k}{1-n_k} \tag{A1}$$

n_k is the usual Fermi function of ε_k, hence

$$\frac{\partial n_k}{\partial T} = -\frac{n_k(1-n_k)}{T}\left[\frac{\partial \varepsilon_k}{\partial T} - \frac{\varepsilon_k}{T}\right]. \tag{A2}$$

Finally, when $\sigma = 0$, ε_k is given by (5) yielding

$$\frac{\partial \varepsilon_k}{\partial T} = \frac{U\,\mathrm{d}n_k}{\mathrm{d}T} \tag{A3}$$

Putting (A2) and (A3) together, we obtain

$$\frac{\partial n_k}{\partial T} = \frac{n_k(1-n_k)\,\varepsilon_k}{1+\dfrac{Un_k(1-n_k)}{T}} \tag{A4}$$

Using the same trick, we can calculate

$$\frac{\partial n_k}{\partial k} = \frac{n_k(1-n_k)\,v_F}{1+\dfrac{Un_k(1-n_k)}{T}} \tag{A5}$$

Hence $\dfrac{\partial n_k}{\partial T}$ and $\dfrac{\partial n_k}{\partial k}$ obey the simple relation

$$\frac{\partial n_k}{\partial T} = -\frac{\varepsilon_k}{Tv_F}\frac{\partial n_k}{\partial k} = \frac{1}{v_F}\frac{\partial n_k}{\partial k}\log\frac{n_k}{1-n_k} \tag{A6}$$

Inserting (A6) into (A1), and noting that

$$\sum_k = \rho v_F \int \mathrm{d}k$$

we finally obtain

$$C_v = -\rho T\int \frac{\partial n}{\partial k}\,\mathrm{d}k\left[\log\frac{n}{1-n}\right]^2 = \frac{\pi^2}{3}\rho T. \tag{A7}$$

The integral reduces to one over dn from 0 to 1, irrespective of its k-dependence : the specific heat is always that of a free Fermi liquid. When $\sigma \neq 0$, the equation for n_k is an *integral* equation. One cannot relate $\partial n_k/\partial T$ and $\partial n_k/\partial k$ simply, and (A7) does not hold : C_v may be more complicated.

Consider now the density response function χ. The single bubble approximation yields

$$\chi_0(\mathbf{q}, \omega) = \sum_{\mathbf{k}} \frac{n_{\mathbf{k}} - n_{\mathbf{k}+\mathbf{q}}}{\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{\mathbf{k}} - \omega} = \sum_{\mathbf{k}} \chi_{0\mathbf{k}} . \tag{A8}$$

If we use for ε_k the Fock quasiparticle energy (3), we must allow for vertex corrections in order to maintain a conserving approximation. The relevant diagrams are shown in figure 13.

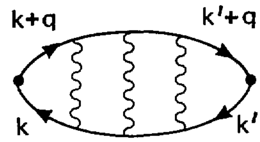

Fig. 13. — A conserving approximation to the density response function χ.

In general, the corresponding Bethe Salpeter equation is an integral equation. In the limit $\sigma = 0$, it reduces to an algebraic equation, with the solution

$$\chi(\mathbf{q}, \omega) = \sum_{\mathbf{k}} \frac{\chi_{0\mathbf{k}}}{1 + U\chi_{0\mathbf{k}}} \tag{A9}$$

(**k** is conserved from one bubble to the next). Using (5), we obtain

$$\chi = \sum_{\mathbf{k}} \frac{n_{\mathbf{k}} - n_{\mathbf{k}+\mathbf{q}}}{\xi_{\mathbf{k}+\mathbf{q}} - \xi_{\mathbf{k}} - \omega} \tag{A10}$$

For small q, we may expend (A10) as

$$\chi = \int \rho \, \mathrm{d}\xi_k \frac{\partial n}{\partial \xi_k} \frac{\mathbf{q} \cdot \mathbf{v}_k}{\mathbf{q} \cdot \mathbf{v}_k - \omega} \tag{A11}$$

The angular integration yields the usual Lindhard function, and the integration over ξ_k gives 1, *irrespective of the form* of $n(\xi_k)$.

Appendix B.

For simplicity, we assume that the density of states and Fermi velocity are constant in the range of interest k_1, k_2. The equation to be solved is then

$$\begin{gathered} \int_{k_1}^{k_2} n(k') \, \mathrm{d}k' \, \mathrm{e}^{-|k-k'|/\sigma} = f(k) \\ f(k) = \frac{2\,\sigma v_{\mathrm{F}}}{U} (k_{\mathrm{F}} - k) - \sigma \, \mathrm{e}^{-(k-k_1)/\sigma} \end{gathered} \tag{B1}$$

(The second term in $f(k)$ comes from the occupancy $n_k = 1$ below k_1). k_{F} is the original Fermi wavevector $\xi_{k_{\mathrm{F}}} = \mu$.

We apply to (B1) the operator $\left[1 - \sigma^2 \frac{\partial^2}{\partial k^2}\right]$. We thus obtain

$$2\,\sigma n_k = f(k) - \sigma^2 f''(k) = \frac{2\,\sigma v_{\mathrm{F}}}{U}(k_{\mathrm{F}} - k) . \tag{B2}$$

Inside the range (k_1, k_2), n_k has the same linear shape as in the case $\sigma = 0$:

$$n_k = \frac{(k_F - k)\, v_F}{U} \tag{B3}$$

Carrying (B3) back into (B1), we adjust k_1 and k_2 so that the integral equation is satisfied. We thus obtain equation (11).

References

[1] KHODEL V. A., SHAGINYAN V. R., *JETP Lett.* **51** (1990) 553, hereafter referred as KS.
[2] DZYALOSHINSKII I. E., LARKIN A. I., *Sov. Phys. JETP* **38** (1974) 202.