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Short Communication

Derivation of relaxational transport equations for a gas of pseudo-Maxwellian molecules

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Abstract. — In this brief note the moment and energy transport equations of relaxational type are derived by the proposed iteration scheme of the solution of the Boltzmann equation for pseudo-Maxwellian molecules.

The Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \int \int (f'f_1' - ff_1)g\sigma(g, \chi) \sin \chi \, d\chi d\mathbf{v}_1$$

for a gas of pseudo-Maxwellian molecules [1] is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = (g_0/4\pi) \int \int f'f_1' \sin \chi \, d\chi d\mathbf{v}_1 - g_0nf,$$  \hspace{1cm} (1)

where $g_0/4\pi = g\sigma(g, \chi) = \text{const}$, $n = \int d^3\mathbf{v} f$.

We shall solve equation (1) iteratively with the use of the following scheme:

$$\frac{\partial f_{k+1}}{\partial t} + g_0nf_{k+1} = (g_0/4\pi) \int \int f_k'(f_k)' \sin \chi \, d\chi d\mathbf{v}_1 - \mathbf{v} \cdot \nabla f_k$$

Similar method was applied in reference [2] to the linearized Boltzmann equation. The first iteration is

$$\frac{\partial f_1}{\partial t} + g_0nf_1 = (g_0/4\pi) \int \int f_0'(f_0)' \sin \chi \, d\chi d\mathbf{v}_1 - \mathbf{v} \cdot \nabla f_0 = g_0nf_0 - \mathbf{v} \cdot \nabla f_0,$$  \hspace{1cm} (2)

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where \( f_0 = n(2\pi kT/m)^{-3/2} \exp\left[-m(v-u)^2/2kT\right]. \) (3)

\( n, u \) and \( T \) here and afterwards are real parameters of the gas. After introduction of \( \varphi = f_1 - f_0 \) we can rewrite equation (2) as follows:

\[
\frac{\partial \varphi}{\partial t} + g_0 \varphi = -\frac{\partial f_0}{\partial t} - \mathbf{v} \cdot \nabla f_0 =
\]

\[
= -f_0 \left[ \frac{1}{n} \frac{dn}{dt} + \mathbf{v} \cdot \mathbf{u} + \frac{mc}{kT} \left( \frac{du}{dt} + \frac{1}{\rho} \nabla p \right) + \left( \frac{mc^2}{2kT} - \frac{3}{2} \right) \left( \frac{dn}{dt} + \frac{2}{3} \nabla \cdot \mathbf{u} \right) + \right.
\]

\[
\left. + \frac{m}{kT} \left( \mathbf{c} - \frac{1}{3} \mathbf{c}^2 \mathbf{I} \right) : \nabla \mathbf{u} + \mathbf{c} \cdot \left( \frac{mc^2}{2kT} - \frac{5}{2} \right) \nabla \ln T \right],
\] (4)

where \( \mathbf{d}/dt = \partial/\partial t + \mathbf{v} \cdot \nabla \), \( \mathbf{c} = \mathbf{v} - \mathbf{u} \).

In order to obtain the relaxation equations for the fluxes we multiply equation (4) by the corresponding polynomial of molecular velocity and integrate over velocities. The right-hand side of the resulting equation is expressed in terms of temporal and spatial derivatives of the parameters of \( f_0 \), i.e. \( n, u, T \). These parameters do not contain any \( \varphi \) contributions and hence obey Euler equations as follows from equation (4). This follows from multiplication of equation (4) by 1, c, and \( c^2 \) and integration over c.

Since the tensor of viscous stresses is

\[
\pi_{\alpha\beta} = m \int c_\alpha c_\beta \varphi \, d^3c = m \int v_\alpha v_\beta \varphi \, d^3v
\]

we can obtain in this way

\[
\frac{\partial \pi_{\alpha\beta}}{\partial t} + g_0 n \pi_{\alpha\beta} = -(\partial/\partial t)(p \delta_{\alpha\beta} + p u_\alpha u_\beta) - \nabla_\gamma (p u_\alpha u_\beta u_\gamma + p u_\gamma \delta_{\alpha\beta}) - \nabla_\beta (p u_\alpha) - \nabla_\alpha (p u_\beta).
\] (5)

Taking into account the Euler equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \rho \frac{d\mathbf{u}}{dt} + \nabla p = 0, \quad \frac{dp}{dt} + (5/3)p \nabla \cdot \mathbf{u} = 0
\]

we can derive from equation (5) that

\[
\frac{\partial \pi_{\alpha\beta}}{\partial t} = -p \varepsilon_{\alpha\beta} - g_0 n \pi_{\alpha\beta}, \quad \varepsilon_{\alpha\beta} = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - (2/3) \delta_{\alpha\beta} \nabla_\gamma u_\gamma.
\] (6)

If the left-hand side is neglected we receive the well-known linear Navier-Stokes law

\[
\pi_{\alpha\beta} = -p \varepsilon_{\alpha\beta}/ng_0 = -\mu \varepsilon_{\alpha\beta}.
\] (6')

The left-hand side describes flows for which the viscous stresses change quickly. Equation (6) has a formal solution

\[
\pi_{\alpha\beta} = -\int_{-\infty}^{t} \exp \left[-g_0 \int_{t'}^{t} n(\tau) d\tau \right] (p \varepsilon_{\alpha\beta})_{t'} \, dt'
\]

Here the subscript \( t' \) shows that the hydrodynamical values in parentheses should be calculated at the moment \( t' \), i.e. earlier than at \( t \). Equation (7) is the so-called transport equation with
delay (or with memory) [3], but contrary to [2, 3] in the present note the memory kernel is expressed explicitly.

We can do similar calculations to derive the relaxational equation for the heat flux

\[ q = \frac{m}{2} \int \left( \frac{5}{2} p u_\alpha + \frac{1}{2} \rho u^2 \right) d^3 v. \]

In order to do this we multiply equation (4) by \( (m/2) v_\alpha v^2 \) and integrate over velocity. Taking into account that

\[ \int (m/2) v_\alpha v^2 \varphi \ d^3 v = q_\alpha + u_\beta \pi_{\alpha\beta} \]

we have the equation similar to the preceding one:

\[
- \left( \frac{\partial}{\partial t} + n_0 \right) (q_\alpha + u_\beta \pi_{\alpha\beta}) = \frac{\partial}{\partial t} \left( \frac{5}{2} p u_\alpha + \frac{1}{2} \rho u^2 u_\alpha \right) + \\
+ \nabla_\beta \left[ \left( \frac{1}{2} \frac{kT}{m} + \frac{1}{2} u^2 \right) \delta_{\alpha\beta} + \left( \frac{7}{2} p + \frac{1}{2} \rho u^2 \right) u_\alpha u_\beta \right].
\]

We simplify the right-hand side of this equation using the Euler equations as earlier:

\[
\frac{\partial q_\alpha}{\partial t} = - \frac{5}{2} p \nabla_\alpha \frac{kT}{m} + \frac{\pi_{\alpha\beta}}{\rho} \nabla_\beta p - g_0 n q_\alpha. \tag{8}
\]

The solution of this equation can be expressed as the transport law with memory similar to equation (7). If we neglect the left-hand side we obtain in the case \( u = 0 \) the classical Fourier law

\[ q_\alpha = - (5p/2g_0 n) \nabla_\alpha (kT/m) = \lambda \nabla_\alpha T. \tag{9} \]

Notice that \( \lambda/\mu c_v = 5/3 \), i.e. the Eucken relation [4] \( \lambda/\mu c_v = 5/2 \) is hence not valid for pseudo-Maxwellian molecules.

Expressions similar to our equations (6), (8) were derived by another approximate method in reference [5], but the equation for \( q_\alpha \) obtained in that paper does not contain the term with \( \pi_{\alpha\beta} \).

Conclusion.

Relaxational equations for momentum and energy fluxes generalizing well-known Navier-Stokes and Fourier laws are obtained using the first iteration of the proposed scheme of solution of the Boltzmann equation. This short note appears to be another proof of the fact that these equations can be obtained from the Boltzmann equation with the same accuracy as the classical ones.

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