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The T and CLP families of triply periodic minimal surfaces. Part 1. Derivation of parametric equations

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Abstract. — The local Weierstrass representation of all members of the T and CLP families of triply periodic minimal surfaces involves integrals of the function $R(\tau) = 1/\sqrt{\tau^8 + \lambda\tau^4 + 1}$ (with $-\infty < \lambda < -2$ for T and $-2 < \lambda < 2$ for CLP), which were previously evaluated only by numerical integration. We show that these integrals are pseudo-hyperelliptic, express them analytically in terms of the incomplete elliptic integral of the first kind, $F(\varphi, k)$, and give explicit general parametric equations for coordinates of these minimal surfaces. The procedure completely obviates the need for numerical integration. The solutions for all three coordinates are intrinsically periodic. The well-known properties of elliptic integrals and their inverse functions provide new insights into the features of triply periodic minimal surfaces, and permit their systematic evaluation.

1. Introduction.

At any point P on an orientable surface there exists a uniquely defined normal vector **n**. A plane containing this vector intersects the surface, forming a plane curve. The curvature of this plane at point P is known as the normal curvature, k_n, of the surface in the tangent direction of the curve. As the plane is rotated about the vector **n**, an infinite set of plane curves are inscribed on the surface, with continuously varying normal curvature at P. The maximum and minimum values of the normal curvature are known as the « principal curvatures », k_1 and k_2, respectively, and tangent directions of the corresponding curves are known as « principal directions ». In general, the principal directions are perpendicular to one another. The mean curvature of the surface at point P is defined as

$$H = \frac{1}{2}(k_1 + k_2)$$

and the Gaussian curvature as

$$K = k_1 k_2 .$$

A minimal surface (MS) is defined as a surface for which the mean curvature is zero at all points [1, 2]. For most points $k_1 = -k_2 \neq 0$. This means that the Gaussian curvature is negative, and that these points are hyperbolic points. The most negative Gaussian curvature occurs at « saddle points ». Exceptionally, there are points with $k_1 = k_2 = 0$ (which corresponds to zero Gaussian curvature) known as « flat points ». A minimal surface given in the Monge form $z = z(x, y)$ must satisfy the Euler-Lagrange equation

$$(1 + z_y^2) z_{xx} - 2\, z_x\, z_y\, z_{xy} + (1 + z_x^2)\, z_{yy} = 0$$

where z_x, z_y and z_z are the first partial derivatives, and z_{xx}, z_{xy} and z_{yy} the second partial derivatives of z, with respect to the variables given in the subscript. For example, the surface $z = \log \dfrac{\cos(x)}{\cos(y)}$ satisfies the above requirement. However, minimal surfaces are very rarely given in the Monge form, and the problem of finding a minimal surface with a given boundary, known as the Plateau Problem, has not been solved in a general case.

If a minimal surface has space-group symmetry, it is periodic in three independent directions. Such surfaces are known as triply periodic minimal surfaces (TPMS). TPMS without self-intersections are of particular interest, both as mathematical objects and because of their relevance to the physical world.

Surfaces in three-dimensional space are usually described by parametric equations $\mathbf{r} = \mathbf{r}(u, v)$, where the components of the position vector $\mathbf{r}$ are functions of two parameters u and v. However, *no analytical expressions for the coordinates of TPMS are known*. Examples of so-called « regular » TPMS [3] are the somewhat arbitrarily named CLP (an acronym for crossed layers of parallels) surfaces discovered by Schwarz [4] and named by Schoen [5] and Schwarz's T (tetragonal distortion of D) surface. The famous Schwarz D (diamond) surface is a special case of the T surface. Some authors, e.g. Mackay [7] give the name F to the D surface, since it has face-centered cubic symmetry. The Schwarz P (primitive cubic symmetry) surface, and Schoen's G (gyroid) surface are related to the D surface by the Bonnet transformation [1]. Before Schoen's 1970 paper, only five regular minimal surfaces (the Neovius surface is not regular) without self-intersections were known : the CLP family, the T family, the H family, the D surface, a member of the T family and the P surface which is adjoint to the D surface. In the past, a surface patch (or surface element, also known as Flächenstück) of such an infinite surface was calculated by numerical integration, and larger portions of the surface were subsequently constructed by combining the patches.

In the last 20 years minimal surface theory has been applied in many areas of the physical and biological sciences [6-14]. Thus Donnay and Pawson [11], and Nissen [12], recognized that the interface between single calcite crystals and amorphous organic matter in the skeletal element in Echinoidea (sea urchins) is described by the P minimal surface. Scriven [13] found that bi-continuous structures of liquid mixtures of water and organics, such as liquid crystals, can be described as periodic minimal surfaces. The relationship between surface descriptions and zeolite structures was recognized by Mackay [14]. Examples of compounds with structures which have been described by minimal surface theory include [6, 8, 10] the zeolites A, N and faujasite, cristobalite, diamond, quartz, ice, W_3Fe_3C (cutting steel), starch and Nb_6F_{15}.

2. Local Weierstrass parametrization of minimal surfaces.

Weierstrass has shown [15] that the Cartesian coordinates (x, y, z) of MS can be *locally* determined by a set of three integrals

$$\begin{aligned} x &= \mathrm{Re} \int_{\omega_0}^{\omega} (1 - \tau^2) R(\tau)\, \mathrm{d}\tau \\ y &= \mathrm{Re} \int_{\omega_0}^{\omega} i (1 + \tau^2) R(\tau)\, \mathrm{d}\tau \\ z &= \mathrm{Re} \int_{\omega_0}^{\omega} 2\, \tau R(\tau)\, \mathrm{d}\tau \end{aligned} \tag{1}$$

where i is the imaginary unit, and $R(\tau)$ some complex function of a complex variable. Thus the Cartesian coordinates of any point on an MS are expressed as the real part (Re) of complex (curvilinear) integrals, evaluated in the complex plane from some fixed point ω_0 to a variable point ω. Integration is carried out within the domain of analyticity of the integrands, and thus the values of the integrals are independent of the path of integration. The value of each integral is the same along any path joining ω_0 and ω (in that order), provided that the path of integration lies entirely within the domain of analyticity. The Weierstrass function $R(\tau)$ completely specifies the local differential geometry of the surface. The Weierstrass equations guarantee that any surface they describe is a minimal surface, but not necessarily free of self-intersections. The problem of finding a minimal surface is thus reduced to solving the integrals (1). So far, analytical solutions, giving expressions for the integrals in (1) have been found only for a handful of minimal surfaces, such as Enneper's and Scherk's surfaces [2].

The T and CLP families of surfaces have been known for more than a century, but the Weierstrass functions for them were discovered only in 1987, when Lidin and Hyde [9, 10] showed that for these surfaces the Weierstrass function has the form

$$R(\tau) = \frac{\kappa}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} \tag{2}$$

where λ is real and κ is, in general, a complex number related to the Bonnet transformation and the normalization constant. They showed that for the T family

$$\lambda = -\left[16\frac{A^2-1}{A^4} - 2\right] \quad \text{with} \quad 0 < A < 1\,.$$

It is easily seen that the above condition is equivalent to $-\infty < \lambda < -2$.

For the CLP family, they found that

$$\lambda = 16(A^2 - A^4) - 2 \quad \text{with} \quad 0 < A < \frac{1}{\sqrt{2}}$$

which is equivalent to $-2 < \lambda < 2$. Schwarz [4] knew the Weierstrass function only for the D surface ($\lambda = -14$). It is interesting to note that no minimal surfaces are known for $\lambda > 2$.

3. Elliptic and hyperelliptic integrals.

Our starting point is the local Weierstrass representation (1) with the Weierstrass function (2) given by Lidin and Hyde [9] for all real values of $\lambda < 2$. With $R(\tau)$ in form (2), the integrals in (1) are hyperelliptic. We note that in non-mathematical literature on minimal surfaces there is much confusion concerning the concept of elliptic and hyperelliptic integrals, as well as the related special functions (incomplete and complete elliptic integrals of the first, the second and the third kind). We therefore begin by discussing these concepts.

A hyperelliptic integral is an integral of the type $\int \mathcal{R}(z, w)\,dz$ where $\mathcal{R}(z, w)$ is a rational function in variables z and w related by an algebraic equation of the type $w^2 = \Phi(z)$ where $\Phi(z)$ is a polynomial of degree greater than four and without multiple roots. Hyperelliptic integrals cannot, in general, be expressed in terms of a finite number of elementary or special functions. In order to evaluate them, it is necessary to use direct numerical integration or complicated series expansions. Hyperelliptic integrals appearing in the description of minimal surfaces have, therefore, always been evaluated numerically. Numerical integration assumes the choice of one of many methods, and special care must be taken of singularities of the integrands. Further, the integration is carried out for points inside an appropriate integration domain, avoiding the singularities, although in some cases important parts of the surface reside very close to the singularities. The coordinates of 30 points of the D, G and P surface elements were found by numerical integration [6].

In very rare cases, some hyperelliptic integrals (known as pseudo-hyperelliptic) can be reduced to integrals involving square roots of polynomials of the third and fourth degree, i.e. to elliptic integrals. Let $\Phi(z) \equiv w^2 = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$ be a polynomial in z with complex coefficients and no multiple roots, and $\mathcal{R}(z, w)$ any rational function in z and w. An integral of the type $\int \mathcal{R}(z, w)\, \mathrm{d}z$, known as an elliptic integral, is not in general expressible in terms of elementary functions only. Any elliptic integral can be expressed as the sum of elementary functions and of the three special functions (canonical forms of elliptic integrals). These canonical forms are incomplete elliptic integrals of the first, the second and the third kind.

We will show that the local Weierstrass parametrization for the T and CLP families of triply periodic minimal surfaces involve pseudo-hyperelliptic integrals. By reducing them, we obtain parametric equations dependent on the value of the coefficient λ, and general expressions describing the T and CLP families of minimal surfaces : i.e. all the known surfaces with the Weierstrass function in form (2). Most importantly, the properties of elliptic integrals of the first kind (one of the so-called « special functions ») are well known, and our results open the way to systematic investigation of the more general features of TPMS.

4. The elliptic integral of the first kind.

Here we will use only the integral in the form

$$\int_{\zeta_0}^{\zeta} \frac{\mathrm{d}z}{w} = \int_{\zeta_0}^{\zeta} \frac{\mathrm{d}z}{\sqrt{a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4}} \tag{3}$$

taken along some rectifiable path in the complex plane, and known as the incomplete elliptic integral of the first kind (IEIFK). By subjecting the variable z to certain algebraic transformations, the function w, and the basic elliptic integrals, can be brought to their normal (standard) forms. There are several normal forms, of which Legendre's $w^2 = (1 - z^2)(1 - k^2 z^2)$ and Jacobi's $w^2 = 1 - k^2 \sin^2 \theta$ normal forms are the most common [16, 17]. Thus, we define the incomplete elliptic integral of the first kind as

$$F(\varphi, k) = \int_0^y \frac{\mathrm{d}t}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \,. \tag{3'}$$

The symbol F is in general use for the incomplete elliptic integral of the first kind, but these is no unanimity as to the way in which the variables are specified. Following Byrd and Friedman [16], we will use the notation $F(\varphi, k)$, and refer to φ as the amplitude of the function, and to k as the modulus. Other notations [18, 19] use the parameter $m = k^2$, or the modular angle $\alpha = \arcsin(k)$, so that the notations $F(\varphi \mid k)$, $F(\varphi \backslash \alpha)$, $F(k ; \varphi)$, $F(k, \varphi)$ and even $F(\varphi)$ are encountered. $F(\varphi, k)$ is usually discussed for $0 < k < 1$ and $0 \leqslant \varphi \leqslant \pi/2$ only, but may be defined for all values (real and complex) of k and φ. In what follows, k is real with $0 < k < 1$ and φ is, in general, a complex number.

The *complete* elliptic integral of the first kind is defined by

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$

so that there is a relation $K(k) = F(\pi/2, k)$.

Elliptic integrals were among the first special (non-elementary) functions to be discovered (in the early 19th century) and, together with elliptic functions, have numerous important applications in various problems of analysis, geometry and physics, especially in mechanics, astronomy and geodesy. Cayley [17] treats the properties of Legendre's integrals in detail, and Byrd and Friedman [16], Milne-Thompson [18] and Tölke [19], give formulae, numerical tables of values, and tables of elliptic integrals. Extensive tables can also be found in Gradshteyn and Ryzhnik [20]. Computation of $F(\varphi, k)$ and $K(k)$ is easy, and will be discussed in detail in another paper. Routines for such computation are available in the NAG, SLATEC and IMSL Fortran libraries.

5. Reduction of the hyperelliptic integrals.

To reduce integrals (1) with $R(\tau)$ in form (2), we introduce three complex functions :

$$\begin{aligned} x_\lambda^*(\omega) &= \Gamma_0 - \Gamma_2 \\ y_\lambda^*(\omega) &= i(\Gamma_0 + \Gamma_2) \\ z_\lambda^*(\omega) &= 2\,\Gamma_1 \end{aligned} \tag{4}$$

where Γ_0, Γ_1 and Γ_2 are

$$\Gamma_0(\omega) = \Gamma_0(\omega_0) + \int_{\omega_0}^{\omega} \frac{\mathrm{d}\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} \tag{5a}$$

$$\Gamma_1(\omega) = \Gamma_1(\omega_0) + \int_{\omega_0}^{\omega} \frac{\tau\,\mathrm{d}\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} \qquad \lambda \in R \tag{5b}$$

$$\Gamma_2(\omega) = \Gamma_2(\omega_0) + \int_{\omega_0}^{\omega} \frac{\tau^2\,\mathrm{d}\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}}\,. \tag{5c}$$

It is clear that the local Weierstrass representation (1) for a particular surface is obtained by taking the real part of the complex functions in (4). The subscripts stress the dependence of the coordinates on a particular value of λ. For $\lambda = \pm 2$ the integrands in (5) are complete squares, and are easily reduced to rational functions, which can always be integrated in terms of a finite number of elementary functions. Thus, for Scherk's surface $\lambda = -2$, and for the adjoint surface $\lambda = 2$. This is in agreement with the result of Lidin and Hyde [9]. Therefore we only need to consider $\lambda \neq \pm 2$, and that branch (sheet) of the square root which takes the value 1 at $\tau = 0$. In this way, the integrands in (5) are always single-valued, provided that during integration we do not cross the branch cut. If this happens, the integrands in (5) have eight singular points (singularities) in the finite complex τ plane. These are shown in figure 1, where α and β are defined as

$$\begin{aligned} \alpha &= 2^{-1/4}\sqrt[4]{-\lambda + \sqrt{\lambda^2 - 4}} \\ \beta &= 2^{-1/4}\sqrt[4]{-\lambda - \sqrt{\lambda^2 - 4}}\,. \end{aligned} \tag{6}$$

It is also clear, from figure 1, that integration (5) is always possible for $|\omega| < m$, where $m = \min(|\alpha|, |\beta|)$. This is in accordance with the usual procedure adopted for the integration domain of the Weierstrass function in form (2), where numerical integration is always carried out within a unit circle inside the appropriate region of the complex plane with four singularities at the boundary. We will reduce the hyperelliptic integrals in (5) to elliptic integrals. With the mapping $\tau' = \tau^2$ (see Tab. I) we obtain :

$$\int_{\omega_0}^{\omega} \frac{d\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} = \frac{1}{2}\int_{\omega_0'}^{\omega} \frac{d\tau'}{\sqrt{\tau'^5 + \lambda\tau'^3 + \tau'}}$$
$$\int_{\omega_0}^{\omega} \frac{\tau\, d\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} = \frac{1}{2}\int_{\omega_0'}^{\omega} \frac{d\tau'}{\sqrt{\tau'^4 + \lambda\tau'^2 + 1}}$$
$$\int_{\omega_0}^{\omega} \frac{\tau^2 d\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} = \frac{1}{2}\int_{\omega_0'}^{\omega} \frac{\tau'\, d\tau'}{\sqrt{\tau'^5 + \lambda\tau'^3 + \tau'}} .$$

It is clear that the mapping reduces the second of the above integrals to an elliptic integral, but the first and third integrals remain hyperelliptic. This means that the expression for the z

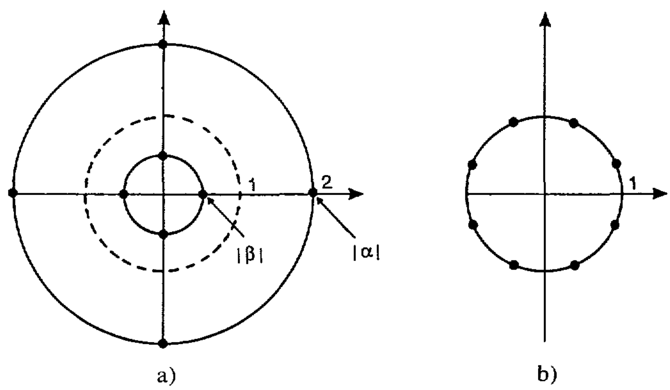

Fig. 1. — The roots (marked with solid points) of the polynomial $\tau^8 + \lambda\tau^4 + 1$ in the complex plane for $\lambda < 2$. The domains of analiticity are : (a) $\lambda < -2$. The roots are given for $\lambda = -14$. (b) $-2 < \lambda < 2$. The roots are given for $\lambda = 0$.

Table I. — *Reduction of pseudo-hyperelliptic integrals to elliptic integrals.*

Mapping	Inverse mapping	Differential	Fixed limit	Variable limit
$\tau' = \tau^2$	$\tau = \sqrt{\tau'}$	$d\tau = \frac{d\tau'}{2\sqrt{\tau'}}$	$\omega_0' = \omega_0^2$	$\omega' = \omega^2$
$\tau'' = \tau' + \frac{1}{\tau'}$	$\tau' = \frac{1}{2}[\tau'' - \sqrt{\tau''^2 - 4}]$	$-\frac{d\tau'}{\tau'\sqrt{\tau'}} = \frac{1}{2}\left[\frac{1}{\sqrt{\tau''+2}} + \frac{1}{\sqrt{\tau''-2}}\right] d\tau''$	$\omega_0'' = \frac{\omega_0'^2 + 1}{\omega_0'}$	$\omega'' = \frac{\omega'^2 + 1}{\omega'}$

coordinate is easily reduced. This result was known to Schwarz [4] for the special case of $\lambda = -14$. With the mapping $\tau'' = \tau' + \frac{1}{\tau'}$ (see Tab. I), which has never been used previously, the first of the above integrals becomes :

$$\frac{1}{2}\int_{\omega_0'}^{\omega} \frac{d\tau'}{\sqrt{\tau'^5 + \lambda\tau'^3 + \tau'}} = \frac{1}{4}\left\{\int_{\omega''}^{\omega_0''} \frac{d\tau''}{\sqrt{(\tau''+2)(\tau''^2 + \lambda - 2)}} + \int_{\omega''}^{\omega_0''} \frac{d\tau''}{\sqrt{(\tau''-2)(\tau''^2 + \lambda - 2)}}\right\}$$

and the third integral becomes :

$$\frac{1}{2}\int_{\omega_0'}^{\omega} \frac{\tau'\, d\tau'}{\sqrt{\tau'^5 + \lambda\tau'^3 + \tau'}} = -\frac{1}{4}\left\{\int_{\omega''}^{\omega_0''} \frac{d\tau''}{\sqrt{(\tau''+2)(\tau''^2 + \lambda - 2)}} - \int_{\omega''}^{\omega_0''} \frac{d\tau''}{\sqrt{(\tau''-2)(\tau''^2 + \lambda - 2)}}\right\}.$$

The result of the two successive mappings in the complex plane on (5a) and (5c), and a single mapping on (5b) is

$$\begin{aligned}
\int_{\omega_0}^{\omega} \frac{d\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} &= \frac{1}{4}\left\{\int_{\omega''}^{\omega_0''} \frac{dt}{\sqrt{(t+2)(t^2 + \lambda - 2)}} + \int_{\omega''}^{\omega_0''} \frac{dt}{\sqrt{(t-2)(t^2 + \lambda - 2)}}\right\} \\
\int_{\omega_0}^{\omega} \frac{\tau\, d\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} &= \frac{1}{2}\int_{\omega_0'}^{\omega'} \frac{dt}{\sqrt{t^4 + \lambda t^2 + 1}} \qquad (7) \\
\int_{\omega_0}^{\omega} \frac{\tau^2 d\tau}{\sqrt{\tau^8 + \lambda\tau^4 + 1}} &= -\frac{1}{4}\left\{\int_{\omega''}^{\omega_0''} \frac{dt}{\sqrt{(t+2)(t^2 + \lambda - 2)}} - \int_{\omega''}^{\omega_0''} \frac{dt}{\sqrt{(t-2)(t^2 + \lambda - 2)}}\right\}
\end{aligned}$$

where ω_0, ω_0' and ω_0'' are the fixed, and ω, ω' and ω'' are the variable integration limits.

From (4), (5) and (7) we obtain only elliptic integrals :

$$x_\lambda^*(\omega) = x_\lambda^*(\omega_0) + \frac{1}{2}\int_{\omega''}^{\omega_0''} \frac{dt}{\sqrt{(t+2)[t - \sqrt{2-\lambda}\,][t + \sqrt{2-\lambda}\,]}} \qquad (8a)$$

$$y_\lambda^*(\omega) = y_\lambda^*(\omega_0) + \frac{i}{2}\int_{\omega''}^{\omega_0''} \frac{dt}{\sqrt{(t-2)[t - \sqrt{2-\lambda}\,][t + \sqrt{2-\lambda}\,]}} \qquad (8b)$$

$$z_\lambda^*(\omega) = z_\lambda^*(\omega_0) + \int_{\omega_0'}^{\omega'} \frac{dt}{\sqrt{t^4 + \lambda t^2 + 1}} \qquad (8c)$$

where

$$\begin{aligned}
x_\lambda^*(\omega_0) &= \Gamma_0(\omega_0) - \Gamma_2(\omega_0) \\
y_\lambda^*(\omega_0) &= i\,[\Gamma_0(\omega_0) + \Gamma_2(\omega_0)] \\
z_\lambda^*(\omega_0) &= 2\,\Gamma_1(\omega_0)
\end{aligned}$$

are constants, and the meaning of ω_0', ω_0'', ω' and ω'' is defined in table I. This demonstrates that equations (5) can be completely reduced to elliptic integrals. All that remains is to express the results in terms of Legendre-Jacobi forms of IEIFK.

6. Evaluation of elliptic integrals.

The results of algebraic reduction of IEIFK in form (3) to the Legendre and Jacobi normal form given in (3′) are listed in tables of integrals where one the limits of integration is the root of the polynomial under the square root, and the other limit is variable [16-18]. It follows that, in order to reduce some particular integral in form (3) to form (3′), it is necessary to find all roots of the polynomial, examine their nature (real or complex), and look up the appropriate integral in the table.

We shall integrate from $\omega_0 = 0$, and consider two distinct cases : $\lambda < -2$ and $-2 < \lambda < 2$. The polynomial can be factorized to $(t-a)(t-b)(t-c)$ for (8a) and (8b), and $(t-a)(t-b)(t-c)(t-d)$ for (8c). A discussion of the number and nature of the roots for both cases is given in table II.

The IEIFK in (8) have been evaluated using integral tables in Byrd and Friedman [16], Milne-Thompson [18] and Gradshteyn and Ryzhnik [20], and checked with Prudnikov [21] and Jahnke and Emde [22]. Analysis of the roots (Tab. II) reveals that six different integrals of three types must be evaluated (see Tab. III).

Integrals (8) expressed in terms of Legendre-Jacobi IEIFK are

$$x_\lambda^* = x_\lambda^*(0) + \frac{1}{2} g_x(\lambda) \,.\, F(\varphi_x(\lambda), k_x(\lambda)) \tag{9a}$$

$$y_\lambda^* = y_\lambda^*(0) + \frac{i}{2} g_y(\lambda) \,.\, F(\varphi_y(\lambda), k_y(\lambda)) \tag{9b}$$

$$z_\lambda^* = z_\lambda^*(0) + g_z(\lambda) \,.\, F(\varphi_z(\lambda), k_z(\lambda)) \tag{9c}$$

where $F(\varphi_i(\lambda), k_i(\lambda))$ are IEIFK with moduli $k_i(\lambda)$ and amplitudes $\varphi_i(\lambda)$ and $i = x, y, z$.

Table II. — *Roots of the polynomials under the square root in equations* (8a)-(8c) *for different values of* λ.

		Roots				
λ	Equation	a	b	c	d	Comments
$\lambda < -2$	(8a)	$\sqrt{2-\lambda}$	-2	$-\sqrt{2-\lambda}$		$\omega'' > a > b > c$
	(8b)	$\sqrt{2-\lambda}$	2	$-\sqrt{2-\lambda}$		$\omega'' > a > b > c$
	(8c)	$\alpha^2 \equiv A$	$\alpha^2 \equiv -A$	$\beta^2 \equiv B$	$-\beta^2 \equiv -B$	polynomial can be factorized to $(t^2-A^2)(t^2-B^2)$ where $A > B$
$-2 < \lambda < 2$	(8a)	$\sqrt{2-\lambda}$	$-\sqrt{2-\lambda}$	-2		$\omega'' > a > b > c$
	(8b)	2	$\sqrt{2-\lambda}$	$-\sqrt{2-\lambda}$		$\omega'' > a > b > c$
	(8c)	$a_x + ia_y$	$a_x - ia_y \equiv \bar{a}$	$-a_x - ia_y$	$-a_x + ia_y \equiv \bar{c}$	$a_x = 0.5(\alpha^2+\beta^2)$ $a_y = -0.5\, i(\alpha^2 - \beta^2)$

Table III. — *Integrals which occur in the evaluation of* (8). *Abbreviations refer to integral numbers in* BF = *Byrd & Friedman* [16] ; MT = *Milne-Thompson* [18] ; GR = *Gradshteyn & Ryzhnik* [20].

Type	Integral	Occurs in (8) in	Comments	Reference
1	$\int_{\omega''}^{\infty} \frac{dt}{\sqrt{(t-a)(t-b)(t-c)}}$	$x_\lambda^*, y_\lambda^*(\lambda < -2)$ $x_\lambda^*, y_\lambda^*(-2 < \lambda < 2)$	$\infty > \omega'' \geqslant a > b > c$	BF 238.00 GR 3.131.8 MT 17.4.65
2	$\int_0^{\omega'} \frac{dt}{\sqrt{(A^2-t^2)(B^2-t^2)}}$	$z_\lambda^*(\lambda < -2)$	$A > B \geqslant \omega' \geqslant 0$	BF 219.00 GR 3.152.7 MT 17.4.45
3	$\int_0^{\omega'} \frac{dt}{\sqrt{t^4+2\mu^2t^2+\nu^2}} = \int_0^{\nu} \frac{dt}{\sqrt{t^4+2\mu^2t^2+\nu^2}} - \int_{\omega'}^{\nu} \frac{dt}{\sqrt{t^4+2\mu^2t^2+\nu^2}}$	$z_\lambda^*(-2 < \lambda < 2)$	$\nu = 1$ $\mu^2 = \frac{1}{2}\lambda$	BF 264.00 GR 3.165.1

Table IV. — *Solutions for x_λ^*, y_λ^* and z_λ^* given in* (9a)-(9c) *for different values of λ.*

		Components of the solution			
λ		$g_x(\lambda), g_y(\lambda), g_z(\lambda)$	$k_x^2(\lambda), k_y^2(\lambda), k_z^2(\lambda)$	$\varphi_x(\lambda), \varphi_y(\lambda), \varphi_z(\lambda)$	Comments
$\lambda < -2$	x_λ^*	$\frac{2}{\sqrt{2}}\frac{1}{\sqrt[4]{2-\lambda}}$	$\frac{1}{2} - \frac{1}{\sqrt{2-\lambda}}$	$\sin^2\varphi_x = \frac{2\sqrt{2-\lambda}}{\omega^2+\omega^{-2}+\sqrt{2-\lambda}}$	$\lvert\omega\rvert < \lvert\beta\rvert$
	y_λ^*	$\frac{2}{\sqrt{2}}\frac{1}{\sqrt[4]{2-\lambda}}$	$\frac{1}{2} + \frac{1}{\sqrt{2-\lambda}}$	$\sin^2\varphi_y = \frac{2\sqrt{2-\lambda}}{\omega^2+\omega^{-2}+\sqrt{2-\lambda}}$	
	z_λ^*	$\frac{\sqrt{2}}{\sqrt{-\lambda+\sqrt{\lambda^2-4}}}$	$\frac{-\lambda-\sqrt{\lambda^2-4}}{-\lambda+\sqrt{\lambda^2-4}}$	$\sin\varphi_z = \frac{\sqrt{2}\,\omega^2}{\sqrt{-\lambda-\sqrt{\lambda^2-4}}}$	
$-2<\lambda<2$	x_λ^*	$\frac{2}{\sqrt{2+\sqrt{2-\lambda}}}$	$\frac{2-\sqrt{2-\lambda}}{2+\sqrt{2-\lambda}}$	$\sin^2\varphi_x = \frac{2+\sqrt{2-\lambda}}{2+\omega^2+\omega^{-2}}$	$\lvert\omega\rvert < 1$
	y_λ^*	$\frac{2}{\sqrt{2+\sqrt{2-\lambda}}}$	$\frac{2\sqrt{2-\lambda}}{2+\sqrt{2-\lambda}}$	$\sin^2\varphi_x = \frac{2+\sqrt{2-\lambda}}{\sqrt{2-\lambda}+\omega^2+\omega^{-2}}$	
	z_λ^*	$\frac{2}{2+\sqrt{2-\lambda}}$	$\frac{8\sqrt{2-\lambda}}{[2+\sqrt{2-\lambda}]^2}$	$\mathrm{tg}\,\varphi_z = \frac{\sqrt{2+\lambda}}{2-\sqrt{2-\lambda}} \cdot \frac{1-\omega^2}{1+\omega^2}$	

For a particular value of $\lambda < 2$ we have :

(a) real functions $g_x(\lambda)$, $g_y(\lambda)$, $g_z(\lambda)$, $k_x(\lambda)$, $k_y(\lambda)$ and $k_z(\lambda)$, listed in table IV, are real constants ;

(b) $x_\lambda^*(0)$, $y_\lambda^*(0)$ and $z_\lambda^*(0)$ are real constants defined in (8) ;

(c) functions $\varphi_x(\lambda)$, $\varphi_y(\lambda)$ and $\varphi_z(\lambda)$, listed in table IV, are complex functions of the complex variable.

7. Conclusions.

(1) For any particular real value of $\lambda < 2$ we obtain Cartesian coordinates (x, y, z) of the minimal surface as ordered sets of values

$$\{\mathrm{Re}\,(x_\lambda^*), \mathrm{Re}\,(y_\lambda^*), \mathrm{Re}\,(z_\lambda^*)\} \tag{10}$$

i.e. real parts (Re) of complex functions x_λ^*, y_λ^* and z_λ^*, defined in (9) and dependent on the complex number ω. It follows that, for any particular value of λ, (9) and (10) are the parametric equations for the corresponding surface.

(2) For $\lambda \in (-\infty, -2)$ we obtain all members of the T family of surfaces, and for $\lambda \in (-2, 2)$ all members of the CLP family. This means that equations (9) and (10) completely describe the T and CLP families of triply periodic minimal surfaces in terms of parametric equations. In this way, the calculation of Cartesian coordinates of any minimal surface belonging to these families reduces to finding the real part of an incomplete elliptic integral of the first kind defined within an appropriate domain in the complex plane.

(3) We note that all three coordinates given by expressions (10) involve the function $F(\varphi, k)$ which is intrinsically periodic [16]

$$F(-\varphi, k) = -F(\varphi, k)$$
$$F(m\pi \pm \varphi, k) = 2m\,.\,K(k) \pm F(\varphi, k)$$

where m is an integer and $K(k)$, the complete elliptic integral, is a constant for a given value of λ. It is thus instantly clear that the surface is indeed triply periodic. This also means that only a small part of any surface need to be calculated, and the coordinates of further points can be derived from the simple periodicity relations given above.

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