# A simple model for shapes of vesicles in two dimensions

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**Abstract.** — The statistical mechanics of vesicles is not yet well understood. We consider a simple stochastic model, the Vesicles-Ising-Droplet-Model, on a two dimensional lattice, to obtain shapes that can be compared to the experimentally observed shapes. We present the model with constant surface and some results, that are in agreement with what is known for real vesicles and we study their phase diagram.

## 1. Introduction.

The study of biological membranes has been an active field of research in the last decade. Several approaches have been tried, in particular the use of statistical physics has been successful.

In nature, a vesicle is a closed lipid bilayer that represents a primitive prototype of a cell. The factors that seem to affect the spectrum of morphological states are: Osmotic pressure, temperature, pH, etc. When some of these factors change so does the vesicle's shape.

Maggs, Leibler, M.E. Fisher and Camacho [2], have developed a model for this problem in two dimensions, in which the membrane is represented by the "pearl necklace model" of polymer theory. They obtained a phase diagram with two variables, osmotic pressure ($\Delta P \leq 0$) and bending rigidity [3]. In particular, they argued that the last variable is the most relevant one for the determination of a vesicle's shape and studied a possible phase diagram, related to the parameters of the model. They found scaling behaviour that relates the area and the radius of gyration of the vesicle to the surface [1, 4, 5]. Mathematical approximations have been made for the case when the boundary of the vesicle is a polygon on a lattice with fixed number of edges enclosing a given area [7]. Other theoretical studies, focussed on mechanical models of vesicles (of Zia *et al.* [6] and Deuling and Helfrich [8]) to explain several three-dimensional vesicle shapes by minimization of the curvature energy of closed membranes.

We develop a new model for vesicle shapes in which the vesicles are defined on a two dimensional lattice and which takes into account: pressure, surface rigidity and surface tension and the corresponding conjugate variables. We consider only the case of vesicle's surface constant. We study two variants: in the first case we have strict area conservation and use

long range Kawasaki dynamics. In the second case the area is controlled by the pressure difference (only $\Delta P < 0$) and a combination of Glauber and Kawasaki dynamics turns out to be appropriate. These Kawasaki long range exchanges distinguish our model from the previously studied tethered models in which only local motions were allowed having as a consequence a very slow dynamics.

The outline of the paper is as follows: in section 2 we describe the model in detail, in section 3 we present the results. Conclusions and discussion of the results are the content of the last section.

## 2. Models.

In this section we discuss a model for the formation of vesicles. For computational simplicity we will study the shape of a vesicle on a square lattice. The controlling factors, such as the bending rigidity and the osmotic pressure will be specified in the dynamics. To this end, we define a Vesicles-Ising-Droplets-Model.

The sites of the lattice can take only two values $1, -1$, i.e. spin up or spin down (occupied or empty), but the vesicles should be compact and so connectivity will be enforced [9, 10]. A vesicle is a cluster of up spins immersed in a sea of down spins, as schematically shown in figure 1.
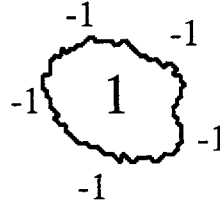


Fig. 1. — Vesicle of up spins in the sea of spin down.

We start the Monte-Carlo simulation, with a randomly choosen connected shape, which contains only up spins as in the figure. We define two models that use combinations of Kawasaki and Glauber dynamics. For the first, we choose randomly two sites with opposite spin values; for the second, one single site, irrespectively of spin direction. The spin flip is realized according to the Boltzmann factor. Each time a site (two sites in Kawasaki dynamics) is considered and three requirements are imposed to make sure that the transformation does not destroy the single connectedness of the surface and the laws conservation.

1. The spin to be flipped should either belong to the boundary of the vesicle, or be a nearest neighbor of a boundary spin outside of the vesicle. In addition, we check that spin flip does not split the vesicle in two or that holes appear in the vesicle, i.e., the cluster should remain compact and singly connected.

2. The sum over the nearest neigbor spins of all the spins to be flipped should be equal to zero. With this we conserve the length of the surface.

3. The sum of the up spins should be constant. With this we conserve the area of the vesicle. With this in mind, we investigated two models.

2.1 VESICLE WITH CONSTANT AREA. — The Hamiltonian for this model is:

$$H = -J \cdot \sum_{i,j} \sigma_i \sigma_j \tag{1}$$

where the summation is carried out over *only* the next nearest neighbours. This Hamiltonian represents the rigidity energy of the vesicle.

In this case we conserve both, the surface (numbers of boundary bonds) and the area (numbers of spins up) of the vesicle. For an update, we choose two sites, one spin up, the other spin down. These sites must be border sites, i.e., have at least one neighbour in each of the two spin states and one must belong to the cluster and the other to the sea of spin down. With this condition, we conserve the area, which can be done with Kawasaki dynamics.

If both sites fulfill the requirements 1-3 the exchange is executed with a probability proportional to the Boltzmann factor $P \sim e^{\frac{-\Delta E}{kT}}$, where $T$ is the temperature, $k$ the Boltzmann factor and $\Delta E$ is the energy difference between the configuration after and before the interchange.

Our model is sensitive to lattice effects. At low temperatures ($kT \leq 1.0$) we observe the formation of facets in diagonal orientation, these facets correspond to the configuration of minimal energy of the Hamiltonian (1). If we increase the temperature ($kT \geq 2.0$) the facets disappear and the border of the vesicle becomes smoother. In three dimensions where one has a finite roughening temperature in the Ising model, this effect should be much stronger.

2.2 VESICLES WITH UNCONSTRAINED AREA. — For this case, the Hamiltonian is:

$$H = -J \cdot \sum_{i,j} \sigma_i \sigma_j + \Delta P \cdot A \tag{2}$$

where the summation is over next nearest neighbours. The first term represents the rigidity measured by the parameter $J$ and in the second term $\Delta P$ is the finite pressure difference, between the interior and the exterior of the vesicle. For this model we use a grand canonical ensemble, while for the first model we worked in a canonical ensemble.

In this case, we use a combination of Kawasaki and Glauber dynamics. A site on the surface of the vesicle is considered and flipped according to the Boltzmann factor only if it fulfills requirements 1 and 2. Every $N$ steps of Glauber dynamics we applied $M$ Kawasaki steps. Initially, only Glauber dynamics was used, typically the dynamics got stuck at tips of branches so that we always found rather rigid branched structures. Moreover, the same shapes of the vesicle were obtained for different values of $\Delta P$. These facts forced us to change the rule and include Kawasaki dynamics. With the imposition of constant surface in Glauber dynamics branches appear, they keep a fixed place in the vesicle and grow in a rigid manner (side branches do not appear). The Kawasaki dynamics perturbes the configuration with changes over long distances, avoiding the formation of localized tips.

## 3. Results.

For each vesicle we measure the components of the inertia tensor and the radius of gyration. We consider each bond between spins of different direction and call the coordinates of its center

$(X_i, Y_i)$. The inertia tensor is defined as

$$\mathbf{I} = \frac{1}{B} \cdot \begin{bmatrix} \sum X_i^2 & \sum X_i Y_i \\ \sum Y_i X_i & \sum Y_i^2 \end{bmatrix}$$

(where $B$ is the number of boundary bonds) and we calculate the eigenvalues $\lambda_1$, $\lambda_2$ of this matrix and their ratio as $\rho = \max(\lambda_1, \lambda_2)/\min(\lambda_1, \lambda_2)$. The radius of gyration is computed through $R_g = \sqrt{\sum(X_i^2 + Y_i^2)/B}$. We performed an average over the different shapes. To do this, each vesicle is translated to the center of the lattice such that the center of mass is the center of the lattice, and rotated into a fixed orientation such that all vesicles share the same principal axis (axis of the largest eigenvalue of the inertia tensor). We superimpose on the same figure various vesicles with the same parameters but different initial configurations as seen in figure 2.



Surface=200

Surface=300
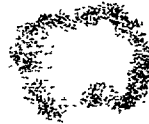
Surface=360

Fig. 2. — Vesicles for area of 900 up spins, $kT = 3.0$.

We made the simulation on a lattice of 500×500 sites. We took 100 different random initial configurations for each model and iterated each vesicle 1.5 million Monte Carlo steps for different values of the surface from 120 to 450. All the simulations were carried out at $kT = 3.0$ for the model with constant area and $kT = 10.0$ for the model with unconstrained area. For the last model, Glauber dynamics was used but every 10000 Monte-Carlo steps, we performed 1000 Monte-Carlo steps with Kawasaki dynamics. We found that, if the area is less than 1300

spins, after this number of steps, the values of the radius of gyration and of the inertia tensor do not change.

We tried to measure other quantities such as the distance from the center of mass to the border of the vesicle parallel to the secondary axis of the shape, as well as the deviation from the shape middle, but these quantities have very big statistical errors.

For both models the calculations are made on the MIMD Alliant FX2800 at GMD having 16 i860 processors in parallel. For the first (constant area) we have 1.2 million updates/sec and for the second 3.1 million updates/sec.

**3.1** VESICLE WITH CONSTANT AREA. — In this case, we conserve the area and the surface of the vesicle. The temperature allows to suppress the effects of the lattice; while not affecting the results. Increasing the temperature above of 2.5, we observe that the influence of the lattice is entirely suppressed. We stop the simulation for the value of area to which the shape of the vesicle is close to a circle.
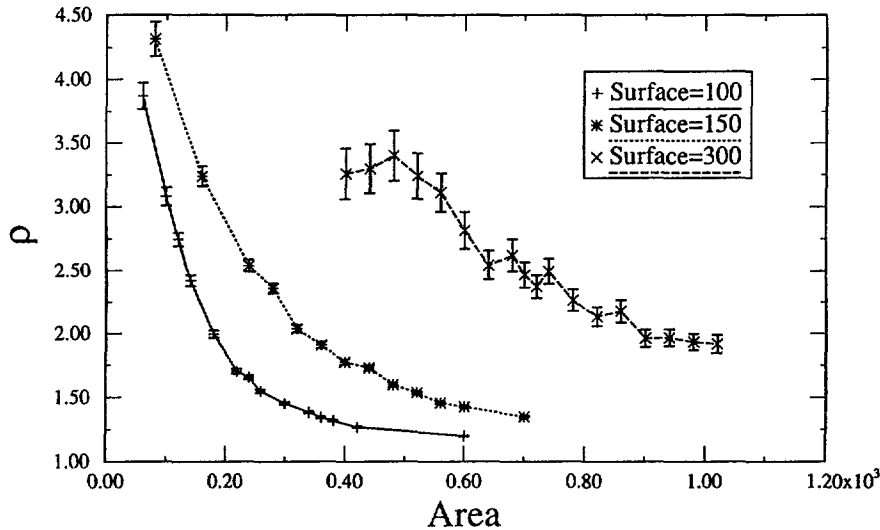


Fig. 3. — Ratio of eigenvalue of inertia tensor *vs.* area.

Some typical shapes are shown in figure 2. We see "vesiculation", i.e., the process in which the vesicle becomes less spherical when the surface increases.

We measure the radius of gyration and the ratio of the eigenvalues of the inertia tensor at fixed surface as function of the vesicle area (Fig. 3). We note that these functions remain smooth as the shape changes from spherical to dumbbell (Fig. 2 with surface of 360) and see no sign of a transition. For large area we obtain the typical figure of minimal energy, a circle, whereby the ratio of the eigenvalues of the inertia tensor is approximately one (Fig. 3). For different surface values and decreasing area the ratio of eigenvalues increases, presenting a behaviour typical for dumbells or ellipses (that are also typical for real vesicles).

Increasing the value of the surface, the shape of the vesicle changes much more slowly and more Monte Carlo steps and higher statistics are necessary.
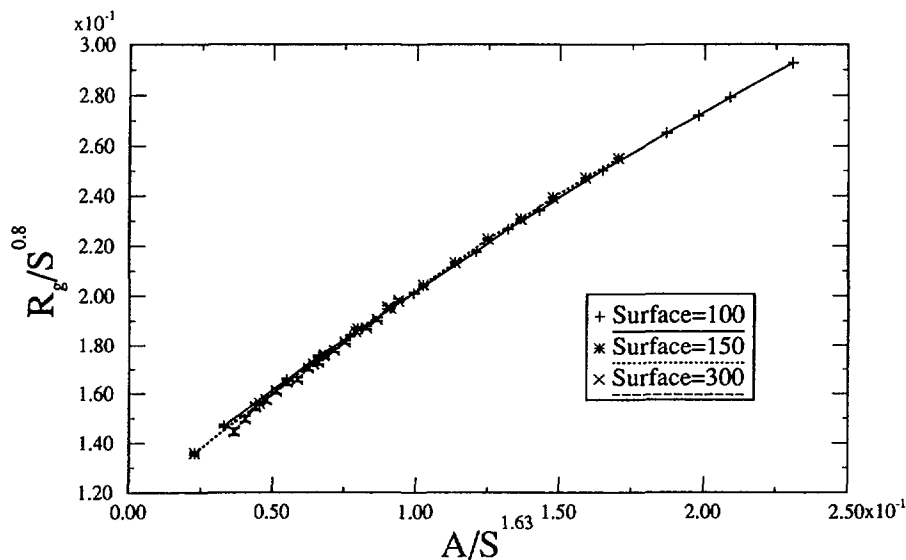
Fig. 4. — Radius of gyration $vs.$ area.

For this simple model we find a power-law that relates the radius of gyration, the area $A$ and the surface $S$, through $R_g \sim S^{0.8} f(AS^{-1.63})$. This behaviour is observed in figure 4, where we tested this hypothesis by a data collaps for different surfaces. This shows that there is an intimate relation between radius of gyration and area. When the surface increases and the area is fixed then we find flaccid vesicles and in the opposite case we find circles.

The computational capacity puts a limit on the possible values of the surface, i.e. when the area is small and the surface is big we need a bigger lattice.

3.2 VESICLE WITH UNCONSTRAINED AREA. — In this case, we measure the same quantities, but we leave the area unconstrained and include a new parameter, the pressure difference. This change to a grand canonical ensemble enhances the lattice problem and the temperature can not suppress it completely, although it is irrelevant as in the first model. As explained above the dynamics is a combination of Kawasaki and Glauber dynamics.

We study a possible phase diagram. We define dimensionless quantities for the parameters, $J/kT$ (related to the rigidity) and $\Delta P/kT$. The second parameter took values in the range $[-0.001, -10.0]$. We also study what happens outside of this range but the shapes of the vesicles do not change very much with the respect to the possibles shapes obtained within this range.

In figure 5, for large $J/kT$ (and small values of $\Delta P/kT$–near zero) the radius of gyration becomes constant. As the value of $J/kT$ decreases, the radius decreases slowly, without a sharp transition, however; as we increase the surface, the curve seems to tend to a limiting value, independent of the surface.

At small values of $\Delta P/kT$ and $J/kT$ the vesicle is ramified and the ratio of the eigenvalues ($\rho$) is about 4.0. When $J/kT$ increases $\rho$ decreases smoothly, and when it is sufficiently high, $\rho$ gets close to unity as in figure 6. For this model, at large rigidity, the shape is a square that is a state of minimal energy of the Hamiltonian (2). The influence of the surface on the value $\rho$ is quite significant ($S \leq 250$). A small change in the surface produces a big change in its
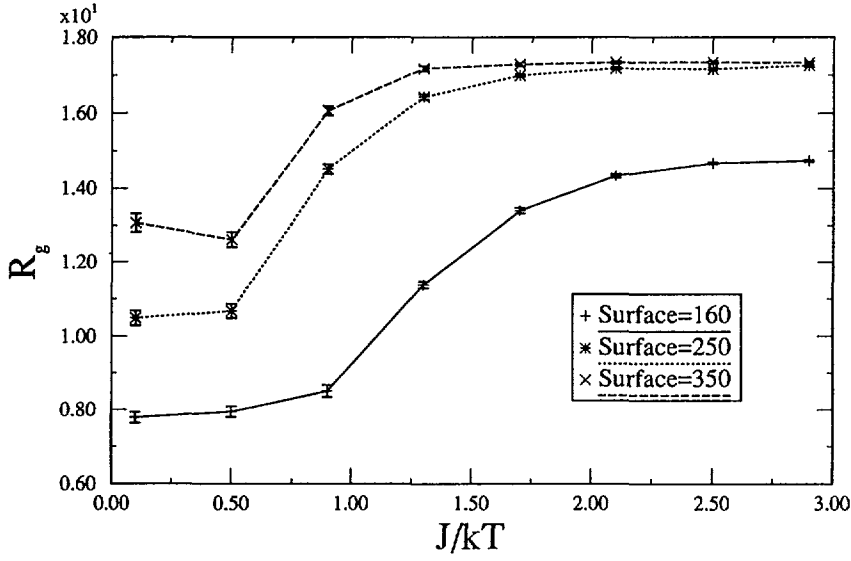
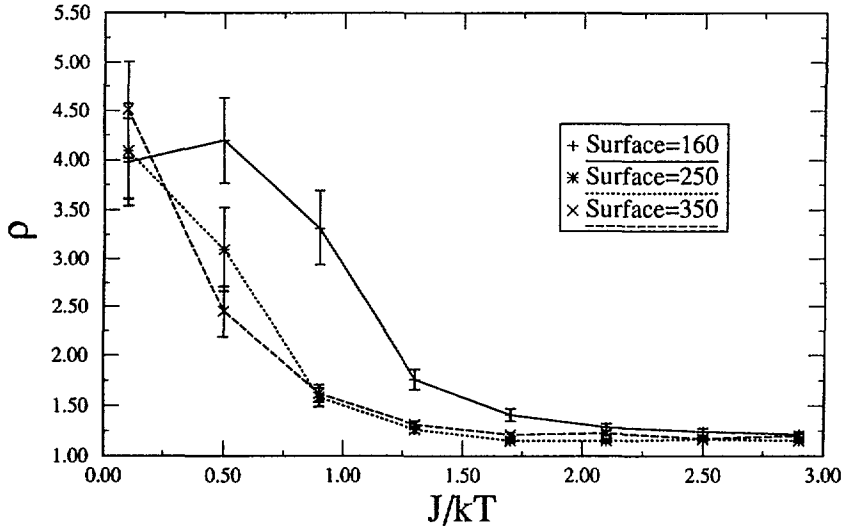Fig. 5. — Radius of gyration *vs.* $J/kT$ with $\Delta P = -0.1$.



Fig. 6. — Ratio of eigenvalue of inertia tensor *vs.* $J/kT$ with $\Delta P = -0.1$.

value. When $S > 250$, we obtain curves similar to the case of small surfaces, but the value of $\rho$ decreases more rapidly. The curve for the surfaces of $S = 250$ and $S = 350$ are close to a curve which may be independent of the surface as in the case of the radius of gyration.

For more negative $\Delta P/kT$ and constant surface the curves of figure 6 and figure 5 (radius of gyration and ratio of eigenvalues *vs.. $J/KT$*) are just translated with respect to the case of small $\Delta P/kT$. In other words, $\Delta P/kT$ has almost no influence on the vesicle. This hypothesis

was also presented by Leibler *et al.* [1] . The surface tension has greater influence over the shape of the vesicle.

## 4. Conclusions and discussion.

We presented a new method for investigating the formation of vesicles in two dimensions. This is an alternative to the "tethered-surface model" proposed by Leibler *et al.*. [1].

We can describe "vesiculation" and obtain realistic shapes of vesicles, in particular in the model with constant area. We found different regimes for the proposed parameters and obtained evidence of scaling behaviour that relates the $R_g$ to the area and the surface. For the model with unconstrained area the parameter that becomes crucial for the changes in the shape is $J/kT$ (bending rigidity).

Both models have slow relaxation, especially in the grand canonical case, where the area is unconstrained and under combined Glauber and Kawasaki dynamics. For this case we have problems with the lattice which do not reduce with the change of the temperature.

We do not observe a sharp phase transition for the quantities that we study, $R_g$ and $\rho$, in agreement with Leibler *et al.* [1] and Camacho and Fisher [3]; we study the fluctuations around the average shape but we do not find evidence for a transition.

We can study other parameters that describe the transition for the different regimes of the vesicle. This model is easily extended to three dimensions and can be instructive to observe the possibles regimes of the shapes and the properties of the vesicle.

## Acknowledgements.

## References

[1] Leibler S., Singh R. and Fisher M.E., *Phys. Rev. Lett.* **59** (18)1989-1992.

[2] Maggs A., Leibler S., Fisher M.E. and Camacho C., The size of an inflated vesicle in two dimensions, preprint.

[3] Camacho C. and Fisher M.E., Computer Simulation Studies in Condensed Matter Physics II, edited by D.P. Landau, K.K. Mon and H.B. Schuttler Eds. (Springer Verlag, Berlin, 1989).

[4] Fisher M.E.,*J. Math. Chem.* **4** (1990)395-399.

[5] Fisher M.E., *Nucl. Phys. B (Proc Suppl)* **5**A (1988) 165-167.

[6] Zia R.K.P., Miao L., Fourcade B., Rao M., Wortis M., Equilibrium budding and vesiculation in the curvature model of fluid lipid vesicles, preprint.

[7] Fisher M.E., Guttmann A. and Whittington S., *J. Phys A* **24** (1991) 3095.

[8] Deuling H. and Helfrich W., *J. Phys. France* **37** (1976) 1335.

[9] Manna S., Herrmann H. and Landau D., *J. Stat. Phys.*, in press (preprint HLRZ, 1991).

[10] Binder K. and Stauffer D., *J. Stat. Phys.* **6** (1972) 49-59.