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A method of integration over matrix variables: IV

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Abstract. — The m-point correlation function

$$\int \left[\prod_{i=1}^{n} \mu(x_i)\right] \left[\prod_{1 \le j < k \le n} |x_j - x_k|^\beta\right] \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n,$$

is calculated for the three values $\beta = 1, 2$ and 4, and integers m and n with $0 \le m \le n$. For some applications one needs this integral when $\mu(x) = \exp[-V(x)]$, $V(x)$ an even polynomial, specially in the limit $n \to \infty$ keeping m finite. A conjecture for this limit in the case $\beta = 2$ is given when $V(x) = x^2 + \gamma x^4$.

1. Introduction and results.

Integrals of the form

$$\int \left[\prod_{k=1}^{n} \mu(x_k)\right] \left[\prod_{1 \le i < j \le n} |x_i - x_j|^\beta\right] \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n, \tag{1.1}$$

have often been encountered in various branches of mathematics and physics. The case $\beta = 2$ was considered first in connection with the orthogonal polynomials and some problems of electrostatics [1]. In some models of mathematical statistics concerning births, deaths, and life expectancies, mathematicians came across the integral (1.1) for the case [2] $\beta = 1$. In algebraic number theory to study the distribution of primes and of the zeros of the Riemann zeta function Selberg [3] needed and evaluated the integral (1.1) with $m = 0$ and arbitrary β, when $\mu(x) = x^a(1-x)^b$, $0 \le x \le 1$. To "explain" the distribution of neutron resonances in nuclear physics Wigner introduced the hypothesis that they are the eigenvalues of a random matrix [4]. This in turn needed

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the integral (1.1) for three values of β, namely 1, 2 and 4 depending on the symmetries of the nuclear system, validity or non-validity of time reversal invariance, integral or half odd integral spin, rotational symmetry or not. Following an argument of 't Hooft [5], certain aspects of some gauge field theories and quantum gravity in two dimensions can be elucidated by studying the integral (1.1) with $\mu(x) = \exp[-V(x)]$, $V(x)$ a polynomial, and taking certain limits [10,11].

The case $\beta = 2$ is the oldest known and the easiest one; for its evaluation one needs to know the properties of determinants and of orthogonal polynomials. For the other two cases $\beta = 4$ and $\beta = 1$, one needs to know the properties of pfaffians and of anti-symmetric scalar products. The treatment of these two cases can be made parallel to the more familiar case $\beta = 2$ if one introduces quaternions and one particular definition (due to Moore) of the determinant of matrices with (non-commuting) quaternion elements. This is much simpler than the usual calculation through the pfaffians.

The evaluation of the integral (1.1) where $\mu(x)$ is a positive weight function with all its moments finite, and $\beta = 1, 2$ or 4, depends on a few theorems given below, after recalling some elementary facts about quaternions.

A quaternion a has the form

$$a = a_0\mathbf{1} + \vec{a}.\vec{\mathrm{e}} \equiv a_0\mathbf{1} + a_1\mathrm{e}_1 + a_2\mathrm{e}_2 + a_3\mathrm{e}_3, \tag{1.2}$$

where a_0, a_1, a_2, a_3 are real or complex numbers, the four units $\mathbf{1}$, e_1, e_2, e_3 satisfy the multiplication rules

$$\mathbf{1}\mathrm{e}_i = \mathrm{e}_i\mathbf{1} = \mathrm{e}_i, \quad \mathrm{e}_i\mathrm{e}_j = -\delta_{ij}\mathbf{1} + \varepsilon_{ijk}\mathrm{e}_k, \quad i,j,k = 1,2,3, \tag{1.3}$$

where δ_{ij}, the Kronecker symbol, is equal to 1 or 0 according as $i = j$ or $i \neq j$, ε_{ijk} is the completely antisymmetric tensor with $\varepsilon_{123} = +1$, and multiplication is associative. The scalar part of a is a_0. The dual of a is $\bar{a} = a_0\mathbf{1} - \vec{a}.\vec{\mathrm{e}}$.

Any quaternion can be represented by a 2×2 matrix with complex elements; for example

$$\mathbf{1} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathrm{e}_1 \rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathrm{e}_2 \rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathrm{e}_3 \rightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{1.4}$$

Conversely, any 2×2 matrix with complex elements can be represented by a quaternion.

In any $N \times N$ matrix $A = [a_{ij}]_N$ with quaternion elements a_{ij}, if we replace a_{ij} by its 2×2 matrix representation, we get a $2N \times 2N$ matrix with complex elements, which we denote $\Theta[A]$. When the a_{ij}'s are complex, the multiplication is commutative, and there is no ambiguity about the determinant of the $N \times N$ matrix $[a_{ij}]_N$ defined in the usual way. However, when the elements a_{ij} are quaternions, the multiplication is no longer commutative, and there is no unique definition of a determinant. We will adopt the following definition [6-9]

$$\det[f_{ij}] = \sum_P (-1)^P \prod_{\text{cycles}} [f_{ab}f_{bc} \cdots f_{da}]^{(0)}, \tag{1.5}$$

where the permutation P consists of the exclusive cycles $(a \rightarrow b \rightarrow c \rightarrow \ldots d \rightarrow a)$, $(-1)^P$ is its sign, and the sum is taken over all $N!$ permutations. The superscript $^{(0)}$ on each cycle means that we take the scalar part of the product.

The dual of the quaternion matrix $A = [a_{ij}]$ is the quaternion matrix $\bar{A} = [\bar{a}_{ji}]$. A self-dual quaternion matrix is such that $a_{ij} = \bar{a}_{ji}$. Let ζ be the antisymmetric matrix with elements

$$\zeta_{2k,2k+1} = -\zeta_{2k+1,2k} = 1, \qquad (k = 0, 1 \ldots), \tag{1.6}$$

and all other elements vanishing. Then

$$\Theta\left[\bar{A}\right] = \zeta\,\Theta[A]^T\,\zeta^{-1} \tag{1.7}$$

Consequently, A is self-dual if and only if $\zeta\,\Theta[A]$ is antisymmetric.

Theorem 1.1.(Dyson) *Let $A = [a_{ij}]_N$ be a $N \times N$ self-dual quaternion matrix, and $\Theta[A]$ be its representation by a $2N \times 2N$ complex matrix. Then*

$$\mathrm{pf}\,\zeta\,\Theta[A] = \det A. \tag{1.8}$$

Here, pf denotes the pfaffian. A first consequence of this theorem is the following formula

$$\det\Theta[A] = (\det A)^2. \tag{1.9}$$

A second consequence is the

Corollary. *Let $A = [a_{ij}]_N$ be a $N \times N$ quaternion matrix, $\bar{A} = [\bar{a}_{ji}]_N$ be its dual, and $\Theta[A]$ be as defined above. Then*

$$\det\Theta[A] = \det\Theta\left[\bar{A}\right] = \det\left(A\bar{A}\right). \tag{1.10}$$

Note that the determinants on the left-hand side of Eqs.(1.9) and (1.10) are of $2N \times 2N$ matrices with complex elements, while those on the right-hand sides of equations (1.8–10) are of $N \times N$ matrices with quaternion elements.

Theorem 1.2. *Let $f(x,y)$ be a function with real, complex or quaternion values, such that*

$$\bar{f}(x,y) = f(y,x), \tag{1.11}$$

where $\bar{f} = f$ if f is real, $\bar{f}$ is the complex conjugate of f if it is complex, and $\bar{f}$ is the dual of f if it is quaternion. Assume that

$$\int f(x,y)f(y,z)\,\mathrm{d}y = f(x,z) + \lambda f(x,z) - f(x,z)\lambda, \tag{1.12}$$

with λ a constant quaternion. Let $[f(x_i,x_j)]_N$ denote the $N \times N$ matrix with its (i,j) element equal to $f(x_i,x_j)$. Then

$$\int \det\left[f(x_i,x_j)\right]_N\,\mathrm{d}x_N = (c - N + 1)\det\left[f(x_i,x_j)\right]_{N-1}, \tag{1.13}$$

where

$$c = \int f(x,x)\,\mathrm{d}x. \tag{1.14}$$

Note that when $f(x,y)$ is real or complex, λ vanishes. Conditions (1.11) and (1.12) then mean that the linear operator defined by the kernel $f(x,y)$ is a projector, and the constant c is its trace (hence a non negative integer).

For a proof of these theorems, see for example reference 9. For a proof of the corollary, see the appendix.

The evaluation of the integral (1.1) depends on expressing its integrand $[\prod_{k=1}^{n}\mu(x_k)]\ |\Delta_n(x)|^{\beta}$, with

$$\Delta_n(x) \equiv \prod_{1\le i<j\le n}(x_i - x_j), \tag{1.15}$$

as $\det\left[f_\beta(x_i, x_j)\right]_n$ where $f_\beta(x,y)$ satisfies equations (1.11) and (1.12). The simplest case is $\beta = 2$, the only one where no quaternions are needed. The next difficult case is $\beta = 4$, where the constant quaternion λ of theorem 1.2 vanishes, and the most difficult one from the mathematical point of view is the case $\beta = 1$. The case $\beta = 2$ is the most widely known and extensively used in the study of matrix models. [10-11] Real symmetric matrices being more familiar than the self-dual quaternion ones, the case $\beta = 1$ received more attention[11] than the case $\beta = 4$. However, the two theorems cited above and the compact form of correlation functions which can be derived from them seems not to have been generally noticed.

The quantities of interest, to be calculated, are the m-point correlation functions given by

$$X_m^{(\beta)}(x_1, .., x_m) \equiv Z_\beta^{-1} \frac{n!}{(n-m)!} \int |\Delta_n(x)|^\beta \left[\prod_{k=1}^{n} \mu(x_k)\right] \mathrm{d}x_{m+1} .. \mathrm{d}x_n, \tag{1.16}$$

where the normalisation constant Z_β is the partition function

$$Z_\beta = \int |\Delta_n(x)|^\beta \left[\prod_{k=1}^{n} \mu(x_k)\right] \mathrm{d}x_1 .. \mathrm{d}x_n. \tag{1.17}$$

When $\beta = 1$, it is convenient to take $2n$ variables instead of n, and to replace the weight $\mu(x)$ by $\sqrt{\mu(x)}$. So, in that case, we use the following definitions

$$X_m^{(1)}(x_1, .., x_m) \equiv Z_1^{-1} \frac{(2n)!}{(2n-m)!} \int |\Delta_{2n}(x)| \left[\prod_{k=1}^{2n} \sqrt{\mu(x_k)}\right] \mathrm{d}x_{m+1} .. \mathrm{d}x_{2n}, \tag{1.18}$$

with

$$Z_1 = \int |\Delta_{2n}(x)| \left[\prod_{k=1}^{2n} \sqrt{\mu(x_k)}\right] \mathrm{d}x_1 .. \mathrm{d}x_{2n}. \tag{1.19}$$

Define three series of monic polynomials, $C_k(x)$, $Q_k(x)$, and $R_k(x)$, of degree k. Recall that a polynomial is called monic when the coefficient of the highest power, i.e. x^k, is one. Let these polynomials satisfy the orthogonality relations

$$\langle C_k, C_l\rangle_C = h_k \delta_{kl}, \tag{1.20}$$
$$\langle Q_k, Q_l\rangle_Q = q_{[k/2]} \zeta_{kl}, \tag{1.21}$$
$$\langle R_k, R_l\rangle_R = r_{[k/2]} \zeta_{kl}, \tag{1.22}$$

where h_k, $q_{[k/2]}$ and $r_{[k/2]}$ are normalisation constants, $[x]$ is the largest integer not greater than x, and where the scalar product $\langle f, g\rangle_C$ and the two *skew* scalar products $\langle f, g\rangle_Q$ and $\langle f, g\rangle_R$ of the functions $f(x)$ and $g(x)$ are defined by

$$\langle f, g\rangle_C \equiv \int_{-\infty}^{+\infty} f(x) g(x)\, \mu(x)\, \mathrm{d}x, \tag{1.23}$$

$$\langle f, g\rangle_Q \equiv \int_{-\infty}^{+\infty} \left[f(x) g'(x) - f'(x) g(x)\right]\, \mu(x)\, \mathrm{d}x, \tag{1.24}$$

$$\langle f, g\rangle_R \equiv \int \int_{-\infty}^{+\infty} f(x) g(y) \sqrt{\mu(x)\mu(y)}\, \varepsilon(y-x)\, \mathrm{d}x \mathrm{d}y, \tag{1.25}$$

with $\varepsilon(x)$ equal to $1/2$ or $-1/2$ according as $x > 0$ or $x < 0$.

Define the quaternions $\chi_k(x)$ and $\varphi_k(x)$, the 2×2 matrix representations of which are

$$\Theta\left[\chi_k(x)\right] \equiv \begin{bmatrix} Q_{2k}(x) & Q_{2k+1}(x) \\ Q'_{2k}(x) & Q'_{2k+1}(x) \end{bmatrix}, \tag{1.26}$$

$$\Theta\left[\varphi_k(x)\right] \equiv \begin{bmatrix} \Phi_{2k}(x) & \Phi_{2k+1}(x) \\ \Phi'_{2k}(x) & \Phi'_{2k+1}(x) \end{bmatrix}, \tag{1.27}$$

with

$$\Phi_k(x) \equiv \int_{-\infty}^{+\infty} \varepsilon(x-y)\, R_k(y)\, \sqrt{\mu(y)}\, \mathrm{d}y. \tag{1.28}$$

Define the real $f_2(x,y)$ and the two quaternions $f_4(x,y)$ and $f_1(x,y)$

$$f_2(x,y) \equiv \sqrt{\mu(x)\mu(y)} \sum_{k=0}^{n-1} \frac{1}{h_k} C_k(x) C_k(y), \tag{1.29}$$

$$f_4(x,y) \equiv \sqrt{\mu(x)\mu(y)} \sum_{k=0}^{n-1} \frac{1}{q_k} \chi_k(x) \bar{\chi}_k(y), \tag{1.30}$$

$$f_1(x,y) \equiv \sum_{k=0}^{n-1} \frac{1}{r_k} \varphi_k(x) \bar{\varphi}_k(y) - \begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix}. \tag{1.31}$$

Then, these three quantities satisfy both conditions (1.11) and (1.12) of theorem 1.2, and our final result is the general formula

$$X_m^{(\beta)}(x_1, .., x_m) = \det\left[f_\beta(x_i, x_j)\right]_m. \tag{1.32}$$

valid for the three values of β.

We recover the known expressions [12] of the partition functions Z_β

$$Z_2 = n! \prod_{k=0}^{n-1} h_k, \qquad Z_4 = n! \prod_{k=0}^{n-1} q_k, \qquad Z_1 = 2^n (2n)! \prod_{k=0}^{n-1} r_k. \tag{1.33}$$

From the positivity of these functions for every n, we deduce that h_k, q_k and r_k are positive numbers for every k.

All these results are valid for any positive weight $\mu(x)$, provided that, as we already said, all its moments are finite, i.e.

$$\int x^j \mu(x)\, \mathrm{d}x < \infty, \quad j = 0, 1, 2, .. \tag{1.34}$$

In particular, the weight $\mu(x) = \exp(-x^2)$ was extensively studied in connection with the theory of random matrices [13], while the cases $\mu(x) = 1$ $(-1 \le x \le 1)$, and $\mu(x) = \mathrm{e}^{-x}$ $(0 \le x < \infty)$, were worked out as examples in an appendix of reference [9].

In section 3, we study the large n behaviour of the correlation functions $X_m^{(\beta)}(x_1, .., x_m)$, with a weight of the form

$$\mu(x) = \mathrm{e}^{-(x^2 + \gamma x^4)}, \quad (-\infty \le x \le +\infty). \tag{1.35}$$

From a reasonable conjecture on the large n behaviour of $C_n(x)$, we deduce that

$$f_2(x,y) \underset{n\to\infty}{\approx} \frac{1}{\pi} \frac{\sin\left[\left(\frac{4n}{3}\right)^{3/4} \gamma^{1/4}(x-y)\right]}{x-y}, \tag{1.36}$$

which together with equation (1.32) gives the asymptotic behaviour of $X_m^{(2)}(x_1, .., x_m)$ for every m.

For some applications a different large n limit is of interest [10, 11, 16]; one takes $\gamma = g/n$, so that the weight function $\mu(x)$ and hence the three sets of polynomials depend on n. If we know the limiting form of $f_\beta(x,y)$, we know the same for all the correlation functions. In the case $\beta = 2$, $f_\beta(x,y)$ is also the two point function, but for $\beta = 4$ or 1 this is not so. Little is known about the limiting correlation functions except for the one point function or the level density [16]

$$X_1^{(\beta)}(x) \equiv \sigma_\beta(x) = \frac{2\sqrt{n}}{\pi\beta}\left(1 + g\frac{c^2}{n} + 2g\frac{x^2}{n}\right)\left(\frac{c^2}{n} - \frac{x^2}{n}\right)^{1/2}, \tag{1.37}$$

where c^2/n is given by

$$c^2 = n\left(\frac{\sqrt{1+6g}-1}{3g}\right). \tag{1.38}$$

This limit will not be considered here.

Another important point not considered here is the following. Knowing the m-point correlation function for each m what observable consequences, if any, can be deduced from the lattice gauge theories, quantum gravity or the superstrings.

2. Proof of equations (1.20-32).

2.1 CASE β=2 — We write $\Delta_n(x)$ as a $n \times n$ determinant

$$\Delta_n(x) = \det\left[x_i^{j-1}\right]_n = \det\left[C_{j-1}(x_i)\right]_n, \tag{2.1}$$

where the $C_j(x)$'s are monic polynomials of degree j. Then

$$\begin{aligned}\Delta_n^2(x) &= \det\left[\sum_{k=0}^{n-1} C_k(x_i)C_k(x_j)\right]_n \\ &= \left(\prod_{k=0}^{n-1} h_k\right) \det\left[\sum_{k=0}^{n-1} \frac{1}{h_k} C_k(x_i)C_k(x_j)\right]_n\end{aligned} \tag{2.2}$$

where $h_0, h_1, .., h_{n-1}$ are non zero constants. Now choose the $C_k(x)$'s as the orthogonal polynomials

$$\int_{-\infty}^{+\infty} C_k(x)C_l(x)\,\mu(x)\mathrm{d}x = h_k\delta_{kl}, \tag{2.3}$$

and set

$$f_2(x,y) \equiv \sqrt{\mu(x)\mu(y)} \sum_{k=0}^{n-1} \frac{1}{h_k} C_k(x)C_k(y). \tag{2.4}$$

Then

$$f_2(x,y) = f_2(y,x) = f_2^*(x,y), \tag{2.5}$$

and from the orthogonality relation (2.3) one has

$$\int_{-\infty}^{+\infty} f_2(x,y) f_2(y,z) \mathrm{d}y = f_2(x,z). \tag{2.6}$$

Thus $f_2(x,y)$ satisfies both conditions (1.11) and (1.12) of theorem 1.2. The constant c defined by equation (1.14) is equal to n, so that equation (1.12) writes

$$\int_{-\infty}^{+\infty} \det\left[f_2(x_i,y_j)\right]_{p+1} \mathrm{d}x_{p+1} = (n-p) \det\left[f_2(x_i,y_j)\right]_p . \tag{2.7}$$

The $m-$point correlation function takes the form

$$X_m^{(2)}(x_1,..,x_m) = Z_2^{-1} \frac{n!}{(n-m)!} \left(\prod_{k=0}^{n-1} h_k\right) \int \mathrm{d}x_{m+1}..\mathrm{d}x_n \det\left[f_2(x_i,y_j)\right]_n \tag{2.8}$$

Successive applications of equation (2.7) allow us to perform the integrations over the variables x_{m+1} to x_n. The calculation of the normalization constant Z_2 along the same lines is straightforward, and the final result is equation (1.32) for $\beta = 2$.

2.2 CASE β=4 — We write $\Delta_n^4(x)$ as a confluent alternant [9]

$$\Delta_n^4(x) = \det\begin{bmatrix} x_i^{j-1} \\ (j-1)x_i^{j-2} \end{bmatrix}_{2n} = \det\begin{bmatrix} Q_{j-1}(x_i) \\ Q'_{j-1}(x_i) \end{bmatrix}_{2n} \tag{2.9}$$

$$(i = 1,2,..,n;\ j = 1,2,..,2n),$$

where the $Q_j(x)$'s are monic polynomials of degree j, and prime denotes the derivation. Define the quaternion $\chi_k(x)$, which for short we identify with its 2×2 matrix representation

$$\chi_k(x) = \begin{bmatrix} Q_{2k}(x) & Q_{2k+1}(x) \\ Q'_{2k}(x) & Q'_{2k+1}(x) \end{bmatrix}. \tag{2.10}$$

Its dual is

$$\bar{\chi}_k(x) = \begin{bmatrix} Q'_{2k+1}(x) & -Q_{2k+1}(x) \\ -Q'_{2k}(x) & Q_{2k}(x) \end{bmatrix}. \tag{2.11}$$

Then $\Delta_n^4(x)$ can be written as follows

$$\Delta_n^4(x) = \det \Theta[Q] \tag{2.12}$$

$$= \det(Q\bar{Q}), \tag{2.13}$$

where Q is the $n \times n$ quaternion matrix with elements

$$Q_{ij} = \chi_{j-1}(x_i). \tag{2.14}$$

To write equation (2.13), use has been made of the corollary of theorem 1.1. Let us set

$$f_4(x,y) \equiv \sqrt{\mu(x)\mu(y)} \sum_{k=0}^{n-1} \frac{1}{q_k} \chi_k(x)\, \bar{\chi}_k(y), \tag{2.15}$$

where the q_k's are non zero constants. Then

$$\left[\prod_{k=1}^{n} \mu(x_k)\right] \det(Q\bar{Q}) = \left(\prod_{k=0}^{n-1} q_k\right) \det\left[f_4(x_i, x_j)\right]_n \tag{2.16}$$

Obviously $f_4(x,y)$ satisfies the first condition, equation (1.11), of theorem 1.2. Let us now see for the second condition, equation (1.12). From equations (2.10) and (2.11) one readily obtains

$$\int_{-\infty}^{+\infty} \bar{\chi}_k(x)\chi_l(x)\,\mu(x)\,\mathrm{d}x = \begin{bmatrix} \langle Q_{2l}, Q_{2k+1}\rangle_Q & \langle Q_{2l+1}, Q_{2k+1}\rangle_Q \\ \langle Q_{2k}, Q_{2l}\rangle_Q & \langle Q_{2k}, Q_{2l+1}\rangle_Q \end{bmatrix}, \tag{2.17}$$

where we have used the skew scalar product $\langle\ ,\ \rangle_Q$ defined in Eq.(1.24). Now choose the $Q_k(x)$'s as the skew orthogonal polynomials satisfying

$$\langle Q_{2k}, Q_{2l+1}\rangle_Q = q_k \delta_{kl} \tag{2.18}$$

$$\langle Q_{2k}, Q_{2l}\rangle_Q = \langle Q_{2l+1}, Q_{2k+1}\rangle_Q = 0. \tag{2.19}$$

Equation (2.17) then becomes

$$\int_{-\infty}^{+\infty} \bar{\chi}_k(x)\chi_l(x)\,\mu(x)\,\mathrm{d}x = q_k \delta_{kl}\, \mathbf{1}, \tag{2.20}$$

(recall that 1 designates the quaternion unity, or the unit 2×2 matrix). As a consequence

$$\int_{-\infty}^{+\infty} f_4(x,y) f_4(y,z)\,\mathrm{d}y = f_4(x,z) \tag{2.21}$$

The end of the calculation goes exactly as in the case $\beta = 2$. Here again the constant c defined by equation (1.14) is equal to n, and the final result is equation (1.32) for $\beta = 4$.

2.3 CASE β=1 — As already noted, it is convenient in this case to take $2n$ variables instead of n, and to change $\mu(x)$ into $\sqrt{\mu(x)}$. The integrand we have to deal with is thus:

$$X = \left[\prod_{k=1}^{2n} \sqrt{\mu(x_k)}\right] |\Delta_{2n}(x)|\,. \tag{2.22}$$

We first note that the sign of the Vandermonde determinant $\Delta_{2n}(x)$ is given [14] by the pfaffian of the $2n \times 2n$ matrix $[\mathrm{sgn}(x_j - x_i)]_{2n}$, so that we can write

$$X = 2^n \left[\prod_{k=1}^{2n} \sqrt{\mu(x_k)}\right] \Delta_{2n}(x)\, \mathrm{pf}\left[\varepsilon(x_j - x_i)\right]_{2n}\,, \tag{2.23}$$

where $\varepsilon(x)$, defined in the previous section, is half the sign function.

Second, we replace the Vandermonde determinant $\Delta_{2n}(x)$ by the determinant of the $2n \times 2n$ matrix $[R_{j-1}(x_i)]_{2n}$, where the $R_k(x)$'s are monic polynomials of degree k. Define the quaternion $\varphi_k(x)$, which for short we identify with its 2×2 matrix representation

$$\varphi_k(x) = \begin{bmatrix} \Phi_{2k}(x) & \Phi_{2k+1}(x) \\ \Phi'_{2k}(x) & \Phi'_{2k+1}(x) \end{bmatrix}, \tag{2.24}$$

where

$$\Phi_k(x) = \int_{-\infty}^{+\infty} \varepsilon(x-y)\, \sqrt{\mu(y)}\, R_k(y)\, \mathrm{d}y. \tag{2.25}$$

Define also the two self-dual quaternions

$$g(x,y) = \sum_{k=0}^{n-1} \frac{1}{r_k} \varphi_k(x)\, \bar{\varphi}_k(y), \tag{2.26}$$

$$f_1(x,y) = g(x,y) - \begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix}. \tag{2.27}$$

where the r_k's are non zero constants. Obviously, the first condition (1.11) of theorem 1.2 is satisfied by both $g(x,y)$ and $f_1(x,y)$.

From the above definitions, one readily obtains

$$\int_{-\infty}^{+\infty} \bar{\varphi}_k(x) \varphi_l(x)\, \mathrm{d}x = 2 \begin{bmatrix} \langle R_{2l}, R_{2k+1}\rangle_R & \langle R_{2l+1}, R_{2k+1}\rangle_R \\ \langle R_{2k}, R_{2l}\rangle_R & \langle R_{2k}, R_{2l+1}\rangle_R \end{bmatrix}, \tag{2.28}$$

where we have used the skew scalar product $\langle\ ,\ \rangle_R$ defined by equation (1.25). Now choose the $R_k(x)$'s as the skew orthogonal polynomials satisfying

$$\langle R_{2k}, R_{2l+1}\rangle_R = r_k \delta_{kl} \tag{2.29}$$

$$\langle R_{2k}, R_{2l}\rangle_R = \langle R_{2l+1}, R_{2k+1}\rangle_R = 0. \tag{2.30}$$

Equation (2.28) becomes

$$\int_{-\infty}^{+\infty} \bar{\varphi}_k(x) \varphi_l(x)\, \mathrm{d}x = 2 r_k \delta_{kl}\, \mathbf{1}, \tag{2.31}$$

which entails that $g(x,y)$ satisfies equation (1.12), up to a factor 2

$$\int_{-\infty}^{+\infty} g(x,y) g(y,z)\, \mathrm{d}y = 2\, g(x,z). \tag{2.32}$$

Furthermore, a straightforward calculation shows that

$$\int_{-\infty}^{+\infty} \begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix} \varphi_k(y)\, \mathrm{d}y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \varphi_k(x), \tag{2.33}$$

$$\int_{-\infty}^{+\infty} \bar{\varphi}_k(y) \begin{bmatrix} 0 & \varepsilon(y-z) \\ 0 & 0 \end{bmatrix} \mathrm{d}y = \bar{\varphi}_k(z) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.34}$$

Putting together all these results, one obtains

$$\int_{-\infty}^{+\infty} f_1(x,y) f_1(y,z)\,\mathrm{d}y = f_1(x,z) + \lambda\, f_1(x,z) - f_1(x,z)\,\lambda, \tag{2.35}$$

where λ is the constant quaternion $\begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$. This is the second condition (1.12) of theorem 1.2. The constant c of the same theorem is easily calculated with the help of Eq.(2.31), and found to be equal to $2n$.

Consider now the $2n \times 2n$ self-dual quaternion matrix $\mathcal{M}$ with elements

$$\mathcal{M}_{ij} = f_1(x_i, x_j). \tag{2.36}$$

Note that $[g(x_i, x_j)]_{2n}$ is the product of two rectangular $2n \times n$ and $n \times 2n$ quaternion matrices, with (i,j) elements equal to $r_{j-1}^{1/2} \varphi_{j-1}(x_i)$ and $r_{i-1}^{1/2} \bar{\varphi}_{i-1}(x_j)$. Its $4n \times 4n$ matrix representation is a rank $2n$ matrix with a vanishing determinant.

Let us be more specific. $\mathcal{M}$ being self-dual, $\zeta\ \Theta[\mathcal{M}]$ is antisymmetric. It is convenient to reorder the rows and columns of $\zeta\ \Theta[\mathcal{M}]$ by writing first the $2n$ odd rows (and columns) before the $2n$ even rows (and columns). Let us call $\Xi[\mathcal{M}]$ the new antisymmetric matrix thus obtained. It takes the form

$$\Xi[\mathcal{M}] = \begin{bmatrix} M & M\alpha^{\mathrm{T}} \\ \alpha M & \alpha M \alpha^{\mathrm{T}} - \mathcal{E} \end{bmatrix}, \tag{2.37}$$

where M, $\mathcal{E}$ and α are $2n \times 2n$ matrices. M and $\mathcal{E}$ are antisymmetric

$$M_{ij} = \sum_{k,l=0}^{2n-1} \frac{1}{r_{[k/2]}}\, \Phi'_k(x_i)\, \zeta_{kl}\, \Phi'_l(x_j), \tag{2.38}$$

$$\mathcal{E}_{ij} = \varepsilon(x_j - x_i), \tag{2.39}$$

and α is defined, for almost all x_i's by

$$\Phi_k(x_i) = -\sum_{j=1}^{2n} \alpha_{ij}\, \Phi'_k(x_j), \qquad (i = 1, 2, .., 2n; k = 0, 1, .., 2n-1). \tag{2.40}$$

One easily finds that

$$\det \Theta[\mathcal{M}] = \det \Xi[\mathcal{M}], \tag{2.41}$$

$$\det \mathcal{M} = \operatorname{pf} \zeta\Theta[\mathcal{M}] = (-1)^n \operatorname{pf} \Xi[\mathcal{M}]. \tag{2.42}$$

Obviously, if we forget the matrix $\mathcal{E}$, the $2n$ last rows (columns) of the matrix (2.37) are linear combinations of the $2n$ first rows (columns). As a result, the determinant of $\Xi[\mathcal{M}]$ is independent of α :

$$\det \Xi[\mathcal{M}] = (\det M)(\det \mathcal{E}). \tag{2.43}$$

Its pfaffian, linear function of the elements of α, indeed is also independent of α, since squared it coincides with the determinant. Thus

$$\operatorname{pf} \Xi[\mathcal{M}] = (-1)^n (\operatorname{pf} M)(\operatorname{pf} \mathcal{E}). \tag{2.44}$$

Next, we remark that the matrix M appears in equation (2.38) as the product $N\,\zeta\,N^{\mathrm{T}}$, where N is the $2n \times 2n$ matrix with (i,k) element equal to $r_{[k/2]}^{-1/2}\,\Phi'_k(x_i)$. Its pfaffian is thus given by

$$\mathrm{pf}\,M = (\det N)\,(\mathrm{pf}\,\zeta), \tag{2.45}$$

$$= \left(\prod_{k=0}^{n-1} \frac{1}{r_k}\right) \left[\prod_{k=1}^{2n} \sqrt{\mu(x_k)}\right] \det[R_{j-1}(x_i)]_{2n}\,. \tag{2.46}$$

Finally, from equations (2.23, 36, 42, 44, 46), we obtain

$$X = 2^n \left(\prod_{k=0}^{n-1} r_k\right) \det[f_1(x_i,x_j)]_{2n}\,. \tag{2.47}$$

Now, the constant Z_1 defined by equation (1.19) can be calculated easily by applying $2n$ times theorem 1.1

$$Z_1 = 2^n \left(\prod_{k=0}^{n-1} r_k\right) \int_{-\infty}^{+\infty} \det[f_1(x_i,x_j)]_{2n}\,\mathrm{d}x_1 .. \mathrm{d}x_{2n}$$

$$= 2^n (2n)! \left(\prod_{k=0}^{n-1} r_k\right). \tag{2.48}$$

The correlation functions defined by equation (1.18) are then given by

$$X_{2n}^{(1)}(x_1, .. , x_{2n}) = \det[f_1(x_i,x_j)]_{2n}\,, \tag{2.49}$$

$$X_m^{(1)}(x_1, .. , x_m) = \frac{(2n)!}{(2n-m)!} \int \det[f_1(x_i,x_j)]_{2n}\,\mathrm{d}x_{m+1} .. \mathrm{d}x_{2n}$$

$$= \det[f_1(x_i,x_j)]_m\,. \tag{2.50}$$

This is equation (1.32) for $\beta = 1$.

3. Large n asymptotics.

It is of particular interest to find the limits, when $n \to \infty$, of the correlation functions $X_m^{(\beta)}(x_1, ..., x_m)$.

We first note that the completeness relations of the orthogonal or skew-orthogonal polynomials $C_n(x)$, $Q_n(x)$ and $R_n(x)$, imply the following large n limits of the $f_\beta(x,y)$'s

$$\lim_{n\to\infty} f_2(x,y) = \delta(x-y), \tag{3.1}$$

$$\lim_{n\to\infty} f_4(x,y) = \frac{1}{2}\begin{bmatrix} \delta(x-y) & \varepsilon(x-y) \\ \delta'(x-y) & \delta(x-y) \end{bmatrix}, \tag{3.2}$$

$$\lim_{n\to\infty} f_1(x,y) = \begin{bmatrix} \delta(x-y) & 0 \\ \delta'(x-y) & \delta(x-y) \end{bmatrix}. \tag{3.3}$$

(Note in the third formula the vanishing of one matrix element: the last term in the right-hand side of equation (1.31), proportional to $\varepsilon(x-y)$, exactly cancels one of the contributions of the previous terms, in the limit $n \to \infty$).

These formulae tell us that the $X_m^{(\beta)}(x_1,...,x_m)$'s diverge when $n \to \infty$, as expected. To know how they diverge, we need more information, which could be provided for example by the large n asymptotic expansions of the polynomials $C_n(x)$, $Q_n(x)$ and $R_n(x)$. When the support of the weight function $\mu(x)$ is compact, such expansions are explicitely known [15], at least for the $C_n(x)$'s. When the support is non compact, the situation is much less comfortable. To our knowledge, explicit formulae are available in the literature only for the classical Laguerre and Hermite polynomials, that is to say respectively, for $\mu(x) = \mathrm{e}^{-x}$ on the positive real axis, and for $\mu(x) = \mathrm{e}^{-x^2}$ on the whole real axis. Indeed, in this last section, we derive as a by-product the leading term of the large n asymptotic expansion of $C_n(x)$, in a non classical case (see Eq.(3.27)).

Here, we are interested in weights of the form

$$\mu(x) = \mathrm{e}^{-V(x)}, \quad (-\infty < x < +\infty), \tag{3.4}$$

where the "potential" $V(x)$ is an even polynomial. The simplest weight beyond the Gaussian one is given by a quartic potential with a positive "coupling constant" γ,

$$V(x) = x^2 + \gamma x^4. \tag{3.5}$$

Since the asymptotic expansions of the corresponding orthogonal polynomials are not available, we shall proceed as follows. It is well known that the functions $X_m^{(\beta)}(x_1,...,x_m)$ are indeed the correlation functions of a classical statistical ensemble of n electrically charged particles (Coulomb gas) in a two dimensional space, constrained to move on a straight line, in the external potential $V(x)$. These functions are calculable in the large n limit by the saddle point method [16]. The corresponding action is

$$W = \sum_{i=1}^{n} V(x_i) - \beta \sum_{1\le i<j\le n} \ln|x_i - x_j|. \tag{3.6}$$

The static equilibrium configuration minimizes this action

$$V'(x_i) - \beta \sum_{j\neq i} \frac{1}{x_i - x_j} = 0. \tag{3.7}$$

In the large n limit, and with a regular potential as described above, it turns out that the charge distribution goes to a continuous function on a finite interval, say $\sigma_\beta(x)$ on $[-c,+c]$, and the above equation becomes

$$V'(x) - \beta\,\mathrm{pv}\int_{-c}^{+c} \frac{\sigma_\beta(y)}{x-y}\,\mathrm{d}y = 0. \tag{3.8}$$

Here pv means that the integral is a principal value integral. Of course, $\sigma_\beta(x)$ is normalized according to

$$\int_{-c}^{+c} \sigma_\beta(x)\,\mathrm{d}x = n. \tag{3.9}$$

With the quartic potential (3.5), the unique solution of the last two equations is found to be

$$\sigma_\beta(x) = \frac{2}{\pi\beta}\left(1+\gamma c^2 + 2\gamma x^2\right)\left(c^2 - x^2\right)^{1/2}, \tag{3.10}$$

with c given by

$$3\gamma c^4 + 2c^2 - 2\beta n = 0. \tag{3.11}$$

When $\gamma = 0$, one recovers the "semi-circle" law of Wigner [17], with an extension c of the charge distribution of order $n^{1/2}$ The situation is radically different when $\gamma > 0$: the distribution (3.10) is no longer a semi-circle, its extension has shrunk and is now of order $n^{1/4}$

$$c \underset{n\to\infty}{\approx} \left(\frac{2\beta n}{3\gamma}\right)^{1/4} \tag{3.12}$$

As expected, the electric charges pack mutually closer in a quartic potential than in a harmonic potential. At large n, the quartic terms in the action W dominate the quadratic ones. This is no longer so, if the coupling constant γ is itself a function of n. With $\gamma = g/n$, the quartic terms are damped at large n, and the extension c remains [16] of order $n^{1/2}$

$$c = n^{1/2}\left[\frac{\sqrt{1+6\beta g}-1}{3g}\right]^{1/2} \tag{3.13}$$

Now, $\sigma_\beta(x)$ is indeed the first term of the large n saddle point expansion of the function $X_1^{(\beta)}(x)$

$$X_1^{(\beta)}(x) = \sigma_\beta(x)\,[1+o(1)]. \tag{3.14}$$

Let us stick to the case $\beta = 2$. Then, thanks to the Christoffel-Darboux [18] formula, the expression of $X_1^{(2)}(x)$ as given by equations (1.29) and (1.32) takes a compact form

$$\begin{aligned} X_1^{(2)}(x) &= \mathrm{e}^{-V(x)}\sum_{i=0}^{n-1}\frac{1}{h_i}C_i^2(x) \\ &= \mathrm{e}^{-V(x)}\frac{1}{h_{n-1}}\left[C_{n-1}(x)C_n'(x) - C_{n-1}'(x)C_n(x)\right]. \end{aligned} \tag{3.15}$$

By analogy with the well known asymptotic formula of Hilb's type for Hermite polynomials (see Eq. (8.22.7) in Ref. [15]), we conjecture that for x finite and n large

$$\mathrm{e}^{-(x^2+\gamma x^4)/2}C_n(x) = a_n\,\left[\cos\left(b_n x - n\pi/2\right) + o(1)\right]. \tag{3.16}$$

Because $C_n(x)$ is monic, has the parity of n, and all its zeros are simple, a_n and b_n are positive. At large n, its density of zeros b_n/π should coincide with the charge density $\sigma_2(x) \approx \gamma c^3/\pi$. With equation (3.12), this gives

$$b_n = \left(\frac{4n}{3}\right)^{3/4}\gamma^{1/4} \tag{3.17}$$

The above conjecture and this value of b_n are corroborated by the fact that $\mathrm{e}^{-V(x)/2}C_n(x)$ satisfies a differential equation [19], which can be shown to take the following form at large n

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + O\left(n^{-1/2}\right)\frac{\mathrm{d}}{\mathrm{d}x} + b_n^2\left[1 + O\left(n^{-1/2}\right)\right]\right\}\mathrm{e}^{-V(x)/2}C_n(x) = 0, \tag{3.18}$$

with precisely the same b_n as in equation (3.17).

Next, we get the expression of a_n by comparing (3.14) and (3.15) at $x = 0$. This gives

$$a_n a_{n-1} = h_{n-1}/\pi, \tag{3.19}$$

or, if we define $\rho_n = h_n/h_{n-1}$,

$$a_n = \rho_{n-1}\, a_{n-2}. \tag{3.20}$$

Now, the ρ_n's satisfy the recurrence relation [12]

$$\rho_n \left[1 + 2\gamma\left(\rho_{n+1} + \rho_n + \rho_{n-1}\right)\right] = n/2, \tag{3.21}$$

from which we deduce their large n behaviour

$$\rho_n = \sqrt{\frac{n}{12\gamma}} \exp\left\{-\frac{1}{\sqrt{12\gamma n}} + O\left(n^{-3/2}\right)\right\}. \tag{3.22}$$

Then, by using Equations (3.20) and (3.22), straightforward calculations lead to

$$a_n = f(\gamma)\left(\frac{n}{12\gamma \mathrm{e}}\right)^{n/4} \mathrm{e}^{-\sqrt{n/12\gamma}}, \tag{3.23}$$

where $f(\gamma)$ is an unknown function of the coupling constant (which cannot be determined by the method used here). Moreover, equation (3.19) gives us the asymptotic behaviour of h_n

$$h_n \approx \pi \mathrm{e}^{1/4} f(\gamma)^2 \left(\frac{n}{12\gamma \mathrm{e}}\right)^{n/2+1/4} \mathrm{e}^{-\sqrt{n/(3\gamma)}} \tag{3.24}$$

We are now in a position to get the first term of the large n expansion of $f_2(x,y)$ at x and y fixed. Thanks to the Christoffel-Darboux [18] formula and equation (3.16), we obtain successively

$$f_2(x,y) = \mathrm{e}^{-[V(x)+V(y)]/2} \frac{1}{h_{n-1}} \left[C_{n-1}(x)C_n(y) - C_{n-1}(y)C_n(x)\right] \tag{3.25}$$

$$\underset{n\to\infty}{\approx} \frac{1}{\pi} \frac{\sin\left[b_n(x-y)\right]}{x-y} \tag{3.26}$$

This last formula (which is in agreement with equation (3.1)), enables us to find the asymptotic behaviour at large n of any correlation function $X_m^{(2)}(x_1, \ldots, x_m)$ given by equation (1.32).

Note that this formula is independent of the unknown function $f(\gamma)$. So is the asymptotic expression of the normalized polynomial $h_n^{-1/2} C_n(x)$

$$h_n^{-1/2} e^{-(x^2+\gamma x^4)/2} C_n(x) \underset{n\to\infty}{\approx} \left(\frac{12\gamma}{n\pi^4}\right)^{1/8} \left\{\cos\left[\left(\frac{4n}{3}\right)^{3/4} \gamma^{1/4} x - n\frac{\pi}{2}\right] + o(1)\right\}. \tag{3.27}$$

Unfortunately, for the cases $\beta = 4$ and 1, there seems to be no compact form of the sums

$$\sum_{k=0}^{n-1} \frac{1}{q_k} \chi_k(x)\bar{\chi}_k(y) \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{1}{r_k} \varphi_k(x)\bar{\varphi}_k(y),$$

analogous to the Christoffel-Darboux formula. So, we cannot reproduce in these cases the above calculation. Use has to be made of the relations [12] between the polynomials $C_n(x)$, $Q_n(x)$ and $R_n(x)$.

Appendix.

Proof of the corollary of theorem (1.1).

Let A be a $N \times N$ quaternion matrix, and $\bar{A}$ be its dual. The $2N \times 2N$ complex matrix representations $\Theta[A]$ and $\Theta\left[\bar{A}\right]$ of A and $\bar{A}$ are related by equation (1.7). Thus

$$\det \Theta[A] = \det \Theta\left[\bar{A}\right], \tag{A1}$$

and

$$(\det \Theta[A])^2 = \det \Theta[A]\,\Theta\left[\bar{A}\right] = \det \Theta\left[A\bar{A}\right] \tag{A2}$$

$$= \left[\det (A\bar{A})\right]^2, \tag{A3}$$

To write the last equality, we have used theorem 1.1, taking advantage of the self-duality of the matrix $A\bar{A}$. Then

$$\det \Theta[A] = \pm \det (A\bar{A}). \tag{A4}$$

Next, we remark that both sides of this equality are continuous functions of the elements of $\Theta[A]$, which excludes one of the two signs $\pm$. It suffices now to take a diagonal matrix A to exclude the minus sign, and this ends the proof of the corollary.

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