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Measurement of spontaneous magnetic fluctuations

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(Reçu le 7 octobre 1986, accepté le 26 janvier 1987)

Abstract. — We describe the first experimental system used to observe the thermal magnetic fluctuations in a magnetic system at equilibrium. We develop the calculation of the coupling of spontaneous magnetic fluctuations to a measuring system, emphasizing the difference with the classical case of susceptibility measurements. We report the power spectrum of the two spin glasses CdCr$_{1.7}$In$_{0.3}$S$_4$ and CsNiFeF$_6$, which exhibit a $1/f$ dependence in the range $10^{-2}$ Hz to $10^2$ Hz. Comparison between this observation and conventional a.c. susceptibility experiments in the same range of frequency supports the applicability of the fluctuation-dissipation theorem for CsNiFeF$_6$.

1. Introduction.

Electrical noise due to the fluctuations of conduction electrons at thermal equilibrium was observed many years ago [1]. The observation gave rise to the theory of the linear irreversible processes whose first form — the famous Nyquist theorem [2] — was generalized later to all linear dissipative systems [3]. This general form is the so called fluctuation dissipation theorem (FDT) [4]. Nevertheless although the FDT states that magnetic fluctuations must exist in real systems, it can be shown easily that they are very difficult to detect experimentally [5]. As a matter of fact, the detection of magnetic fluctuations has been made possible [5-8] by the use of SQUID magnetometers whose sensitivity is several orders of magnitude better than that of classical systems. On the other hand there is, at the day, a renewal of interest for thermodynamic fluctuation studies on magnetic materials, since they can give informations on the dynamics of systems in the absence of any external disturbance. This is of paramount importance in disordered systems, like spin-glasses, where non linearities with field and ergodicity breaking are expected below the transition temperature [9]. Furthermore, fluctuation measurements are a good tool to test the validity of the FDT in such non classical systems.

This paper is devoted to the description of an experimental set-up which has been already used to detect magnetic fluctuations on several insulating spin glasses [5, 6]. Particularly we derive the experimental parameters allowing to perform a quantitative analysis of the measured fluctuations.

Finally, it is worth noting that we are only interested here on equilibrium fluctuations, not on induced effects, among which, for example, is the so called « Barkhausen noise ».


The measurements are performed by an inductive method. A piece of magnetic material is inserted into a pick up coil connected to a detecting system. At the day the best sensitivity is obtained by using a superconducting flux transformer followed by a SQUID system. Thus the experimental set up is essentially a magnetic flux measuring system with high sensitivity. However, the way to relate the mean square magnetization in the material to the mean square flux in the pick up coil is not classical. As will be seen below, the result depends strongly on
the spatial correlations of the magnetic fluctuations, giving rise to two very different behaviors in the two extreme cases of short range and long range correlations (the characteristic length is obviously the characteristic size of the pick up coil).

In the experiments described in the following sections, the correlation length is small. The case of long range correlations is analysed in order to make a comparison with classical susceptibility measurements.

2.1 Relations Between Magnetic Moment Fluctuations and Flux Fluctuations. — The relationship between the magnetization \( \mathbf{m}(r, t) \) — defined as the spatial density of magnetic moments \( \mu \) — and the induced flux \( \phi \) in a pick up coil can be derived from a classical reciprocity theorem of magnetostatics. A volume element \( dr^3 \) located at \( r \) with magnetization \( \mathbf{m}(r, t) \) will induce in the coil a contribution to the flux

\[
d\phi(r, t) = \mathbf{m}(r, t) \cdot \mathbf{h}(r, t) \, dr^3 \tag{1}
\]

where \( \mathbf{h}(r) \) is the magnetic field that the coil would produce if it carried a unit of current (see Fig. 1). Then by integrating over the volume \( V \) of the sample, one obtains

\[
\phi(t) = \int_V \mathbf{m}(r, t) \cdot \mathbf{h}(r) \, dr^3. \tag{2}
\]

For the sake of simplicity, some assumptions are useful:
— the fluctuations of the magnetization are stationary (so is the flux in the coil)
— the fluctuations of the magnetization are homogeneous, isotropic and its components are statistically independent.

Consequently:

\[
\langle m_\mu(r, t) m_\nu(r', t + \tau) \rangle = \langle m(0, 0) m(r - r', \tau) \rangle \delta_{\nu\mu} \tag{4}
\]

Use of equation (3) and of the Wiener-Khintchine theorem yields:

\[
\bar{\phi}^2(\omega) = \int_V \int_V m^2(r - r', \omega) \times h_\mu(r) h_\nu(r') \, dr^3 \tag{6}
\]

where \( \bar{\phi}^2(\omega) \) is the spectral density of the flux in the pick-up coil and \( \bar{m}^2(r - r', \omega) \) is the spectral density of the spatial correlations of magnetization along one arbitrary axis. Equation (6) shows that the coupling between the sample and the pick-up coil depends strongly on the spatial correlations. Actually, there are two extreme cases of interest: the limit of long and short correlation lengths.

One can solve equation (6) for the two extreme cases:

a) Case 1: the fluctuations are considered as spatially uncorrelated. Thus:

\[
\bar{m}^2(r - r', \omega) = \bar{M}^2(\omega) V \delta(r - r') \tag{7}
\]

where \( \bar{M}^2(\omega) \) is the spectral density of the magnetization \( M(t) \) along one arbitrary axis in the volume \( V \)

\[
M(t) = \frac{1}{V} \int_V m(r, t) \, dr^3
\]

\( M(t) \) is an intensive quantity and in general is different from the local magnetization \( m(r, t) \). Equation (6) becomes

\[
\bar{\phi}^2(\omega) = \bar{M}^2(\omega) A_s \tag{8}
\]

with

\[
A_s = V \int_V \sum_\mu h_\mu^2(r) \, dr^3 = V \int_V |h(r)|^2 \, dr^3. \tag{9}
\]

b) Case 2: the correlations are considered to extend over large distances as compared to the size of the coil:

\[
\bar{m}^2(r - r', \omega) = \bar{M}^2(\omega) \tag{10}
\]

and equation (6) becomes

\[
\bar{\phi}^2(\omega) = \bar{M}^2(\omega) A_l \tag{11}
\]
with \[ A_1 = \sum_\mu \left[ \int_V h_\mu(r) \, dr \right]^2. \] (12)

It is important to note that in equations (8) and (11) \( \tilde{M}^2(\omega) \) is the spectral density of the total magnetic moment of the sample divided by the square of the volume. Relations (8) and (11) show that in both cases the flux fluctuations are proportional to the magnetization fluctuations, but the coefficients of proportionality are quite different. Obviously for very large correlation lengths, one finds the relation which corresponds to a classical magnetization measurement. The integral of \( |h(r)|^2 \) appearing in (9), if extended over the whole space, is proportional to the inductance of the measuring coil.

We can characterize the integral (9) as « incoherent » (incoherent relatively to the direction of the field), and note that it gives the energy of the magnetic field \( \mathbf{h} \) in the volume \( V \). On the opposite the integral (12) could be considered as « coherent » since algebraic compensation (by symmetry) may appear in the volume integral.

2.2 THE USE OF THE FLUCTUATION-DISSIPATION THEOREM. — In classical cases, a relation exists between the fluctuations of physical quantities at thermal equilibrium and the deterministic response of the system to their conjugate field. This relation is the so called fluctuation-dissipation theorem. The conditions for its applicability are: i) the response of the medium must be ergodic and linear; ii) the medium must be stationary.

From now on the c.g.s.e.m. system of units will be used. If the susceptibility is defined by

\[ M(\omega) = \chi(\omega) h(\omega) \] (13)

where \( M(\omega) \) is the response of the magnetization to a periodic uniform magnetic field, then provided \( h_\omega \ll k_B T \), the fluctuation-dissipation theorem can be written:

\[ \tilde{M}^2(\omega) = \frac{2 k_B T \chi''(\omega)}{\pi V \omega} \] (14)

where \( T \) is the temperature, \( k_B \) the Boltzman constant and \( \chi'' \) is the imaginary part of \( \chi \).

In the limit of short range correlations, equation (8) leads to

\[ \tilde{\phi}^2(\omega) = A_s \frac{2 k_B T \chi''(\omega)}{\pi V \omega}. \] (15)

This result can be directly obtained. When a piece of material with susceptibility

\[ \chi(\omega) = \chi'(\omega) - i \chi''(\omega) \]

is inserted into a coil with free inductance \( L_0 \), the inductance will change to

\[ L = L_0(1 + 4 \pi q \chi(\omega)). \] (16)

where \( q \) is a filling factor. From the impedance of the coil (its dissipative part is \( 4 \pi q L_0 \omega \chi'' \)) and using the Nyquist theorem one finds:

\[ \tilde{\phi}^2(\omega) = \frac{2 k_B T}{\pi} \frac{\chi''(\omega)}{\omega} 4 \pi L_0 q. \] (17)

Remember that

\[ L_0 = \frac{1}{4 \pi} \int_\infty |h(r)|^2 \, dr \] (18)

(\text{where the integration is performed over the whole space outside the coil wire}). Thus (8), (9), (17) yield

\[ q_s = \frac{\int_V |h(r)|^2 \, dr}{\int_\infty |h(r)|^2 \, dr}. \] (19)

The meaning of equation (19) is that the filling factor \( q \) is the ratio between the « operational » — i.e. pondered by \( |h(r)|^2 \) — volume occupied by the sample and the total « operational » volume.

In the case of short range correlations

\[ \tilde{\phi}^2(\omega) = \tilde{M}^2(\omega) 4 \pi L_0 q_s V \] (20)

with expression (19) for \( q_s \). By generalization, for the case of long range correlations

\[ \tilde{\phi}^2(\omega) = \tilde{M}^2(\omega) V 4 \pi L_0 q_l \] (21)

with

\[ q_l = \frac{1}{V} \int_V |h(r)|^2 \, dr. \] (22)

2.3 THE COUPLING TO THE SQUID SYSTEM. — The experimental flux \( \phi(\omega) \) is measured by coupling the pick up coil to a SQUID system. The pick up coil \( L_0 \), the coupling coil to the SQUID \( L_s \), and the leads connecting them \( l_0 \), form a closed superconducting circuit called the flux transformer (see Fig. 2). The flux transfer ratio is given by the relation:

\[ \xi = \frac{k}{L_0 + L_s + l_e}. \] (23)

Fig. 2. — Schematic representation of the coupling system between the sample and the SQUID.
Where $l_s$ is the self inductance of the SQUID and $k$ is the coupling coefficient between the coil $L_0$ and the SQUID (i.e. their mutual inductance is $k \sqrt{l_s L_s}$). Neglecting $l_e$ which is minimized by twisting the connecting leads, $\xi$ is maximum when $L_0 = L_s$ and 

$$\xi_{\text{max}} = \frac{k}{2} \sqrt{\frac{l_s}{L_0}} \quad (24)$$

then using equation (20) or (22) depending on the case of interest one finds:

$$\bar{\phi}^2(\omega)_{\text{SQUID}} = \xi_{\text{max}}^2 \bar{\phi}^2(\omega) = M^2(\omega) V k^2 l_s \pi q_i \quad i = s \text{ or } l. \quad (25)$$

For a given sample the coupling between the SQUID and the magnetization depends on $L_0$ only via the filling factor $q$.

### 2.4 COMPUTED VALUES OF THE FILLING FACTOR.

Relation (25) can be written

$$\bar{\phi}^2(\omega)_{\text{SQUID}} = \xi_{\text{max}}^2 \bar{\phi}^2(\omega) = M^2(\omega) \frac{k^2 l_s}{4 L_0} V a F_i \quad i = s \text{ or } l \quad (26)$$

where $a$ is the radius of the pick up coil and $F_i$ is given by the following relations:

a) **case of short range correlations**

$$a F_s = \int_V |h(r)|^2 \, dr^3 \quad \text{(see Eq. (19))}$$

b) **case of long range correlations**

$$a F_l = \frac{1}{V} \sum \left( \int_V h_\mu(r) \, dr^3 \right)^2 \quad \text{(see Eq. (22))}$$

We present the numerical results in the case where the pick up coil is a thin ring of radius $a$. (The details of the calculations are given in Appendix A). The sample is a cylinder, centred in the pick up coil, with radius $a - e$ and height $2 A$ (see Fig. 1). In figures 3 and 4, $F_s$ and $F_l$ are plotted as a function of $\Lambda/a$. When $\Lambda/a$ is small ($\Lambda/a \ll e/a$) the magnetic field is approximately constant within the sample, and the filling factors are equal in both cases. Moreover in this range $q$ increases linearly with the volume, thus with $\Lambda/a$. The filling factor deviates from the linearity since the modulus of the magnetic field decreases markedly as the distance from the plane of the ring increases. For short range correlations the filling factor reaches a $a/e$ dependent constant value in the $\Lambda/a \gg 1$ region. In contrast $q_l$ decreases as $(\Lambda/a)^{-1/2}$ in the same region. The difference between the behaviour of the filling factors (or $F$) is enhanced in figure 5. It is not surprising that $F_s$ increases to infinity with $a/e$ (or $\frac{d-e}{a}$) like $\log (a/e)$. It has been seen that the integral of $|h(r)|^2$ defines also the self inductance when taken as follows. Indeed, as we have mentioned in the end of 2.1, in the case of long range correlations, the flux in the coil is due to a spatially constant magnetization and only the contribution of the longitudinal magnetization $m_z$ does not cancel by symmetry. On the other hand for short range correlations, the fluctuations are independent and there is no compensation by symmetry. The variations of $F_s$ and $F_l$ as a function of the relative radius $\frac{a-e}{a}$ for $\Lambda/a = 2$ are presented in figure 5. It is not surprising that $F_s$ increases to infinity with $a/e$ (or $\frac{d-e}{a}$) like $\log (a/e)$. It has been seen that the integral of $|h(r)|^2$ defines also the self inductance when taken
over the whole space, and it is well known that the self inductance for a ring rises to infinity (like \( \log(a/e) \)) when the wire radius decreases towards zero. In this limit of large \( a/e \), the asymptotic filling factor which can be obtained (i.e. for a close winding with a wire of radius \( e \)) is \( q = 0.55 \).

2.5 DISCUSSION. — In our experiments, powdered samples were used. The characteristic length of the grain is small compared to the size of the measuring coil. Thus, in the case of spontaneous magnetic fluctuations, the correlation length can not be longer than the size of the grain and we are obviously in the case of short range correlations, the coupling factor being \( q_s \). However, if a susceptibility experiment is performed with such a sample with a uniform applied magnetic field, the field induced correlation of the magnetization will extend over the whole sample, and thus the coupling factor will be \( q_l \). This would be also the case for fluctuations induced by a spurious external magnetic field; it is worth noting that such a contribution is cancelled in our experiment by the gradiometric geometry of the pick up coil (see Sect. 3).

Since the spontaneous fluctuations of the total magnetization are very small, the coupling with the coil must be drastically optimized. The preceding results show that only the part of the sample near the coil is efficient in the coupling. Contrary to the case of classical susceptibility measurements — i.e. long range correlation — the filling factor \( q_s \) increases significantly when the sample is brought near the coil. Thus in spontaneous magnetic fluctuation experiments the coil must be wound as close as possible to the sample.

3. Experimental

3.1 EXPERIMENTAL SET-UP. — The measurements are performed at 4.2 K in a \(^4\)He dewar. Two mumetal shields (one external at 300 K and one in the \(^4\)He vessel at 4.2 K) and a lead shield screen the experimental space against the ambient magnetic field (see Fig. 6). In order to reduce the sensitivity of the measuring system to the residual ambient magnetic noise — which will induce uniform magnetic fluctuations in the sample — the pick up coil is a third order cylindrical gradiometer, made of \( +3, -6, +6, -3 \) turns of niobium wire, with the following size: coil diameter = 7.2 mm, wire diameter = 0.1 mm, basic spacing between windings = 5 mm. The rod shaped sample is contained in a sample holder of size \( L = 30 \) mm, \( d = 6.5 \) mm. The sample holder can be moved manually from the outside in order to allow noise recordings with the sample in or out of the pick up coils.

The R.F. SQUID is a type MS03 from the LETI (C.E.N.G. Grenoble, France). It exhibits white noise with optimal r.m.s. value \( 10^{-4} \Phi_0 / \sqrt{\text{Hz}} \) (where \( \Phi_0 \) is the flux quantum: \( \Phi_0 = 2.07 \times 10^{-7} \text{gauss/cm}^2 \)) at frequencies down to 0.5 Hz, with a cross-over to \( 1/f \) noise below 0.1 Hz [10]. The SQUID coupling coil is made out of 0.1 mm niobium wire.

Finally, the output voltage is transferred to a 512 channels digital spectrum analyser covering the range \( 10^{-2} \text{Hz} \) to 25 kHz. The analysed amplitudes are converted into r.m.s. flux threading the SQUID \( \Phi_s(\omega) \), by means of the voltage/flux transfer ratio of the SQUID which is \( 8V/\Phi_0 \). The measuring pro-
procedure is the following: a record of the noise spectrum of the empty system is first performed, then the sample is raised into the gradiometer and the noise spectrum is again recorded. The noise spectrum of the sample is the difference between the spectra with the sample in and out of the coil.

As mentioned above, the pick up coil is a third order cylindrical gradiometer, but whatever the considered loop of this coil, the length of the sample allows to consider the asymptotic value of \( F_s(L) \) as a good approximation. The ratio \( a/e \) is estimated to be \( 8 \pm 1 \). Thus for a single loop the factor \( F_s(L) \) is \( 180 \text{ emu} \pm 20 \). The spectral density of the flux is (Eq. (20))

\[
\tilde{\phi}_1^2(\omega) = 4 \pi L_0 V q_s \tilde{M}^2(\omega) = aVF_s(L) \tilde{M}^2(\omega) .
\]  

(27)

Then for a coil of \( N \) turns the flux is proportional to the effective surface:

\[
\tilde{\phi}_{\text{hl}}^2(\omega) = N^2 \tilde{\phi}_1^2(\omega) = aVN^2 F_s(L) \tilde{M}^2(\omega) .
\]  

(28)

The total self inductance \( L_0 \) of the gradiometer has been calculated, taking into account the mutual self inductance between adjacent windings \( (M_{3-6} = 17 \text{ nH} \) between three turns and six turns windings, and \( M_{6-6} = 26 \text{ nH} \) between both six turns windings) : \( L_0 = 1150 \text{ nH} \pm 17\% \). The low values of the mutual inductances between windings show that the respective fluxes are independent. Thus the total flux in the coil is

\[
\tilde{\phi}^2(\omega) = \sum_i N_i^2 \tilde{\phi}_i^2(\omega) = aV \sum_i N_i^2 F_s(L) \tilde{M}^2(\omega) = \sum_i N_i^2 aVF_s(L) \tilde{M}^2(\omega)
\]  

(29)

where the sum is over the four windings. For our gradiometer,

\[
A = (5.9 \pm 0.2) \cdot 10^3 \quad \text{and} \quad q = 0.4 \pm 0.02 .
\]

Finally, the characteristics of the flux transformer are the following: coupling coil \( L_s = 930 \pm 20 \text{ nH} \), input mutual inductance of the twisted pair \( l_e = 30 \pm 5 \text{ nH} \). Thus the flux transfer ratio is derived from (23), \( \xi = (5 \pm 1) \cdot 10^{-3} \).

3.2 RESULTS. — In this section we report the direct observation of thermodynamic magnetic fluctuations in the two spin glasses \( \text{CsNiFeF}_6 \) and \( \text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4 \). The choice of these materials has been determined from the following criteria:

— High value of \( \chi'' \) in the frequency range between \( 10^{-2} \) and \( 10^2 \text{ Hz} \) at 4.2 K.

— No contributions from eddy current noise as for metallic spin glasses

\( \text{CsNiFeF}_6 \) is a crystalline insulator belonging to the modified pyrochlore family. The crystal structure is fcc and the magnetic ions \( \text{Ni}^{2+} \) and \( \text{Fe}^{3+} \) are distributed at random on a network of corner-sharing tetrahedra with near neighbour antiferromagnetic interactions [11]. \( \text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4 \), also an insulating compound, is a dilute spinel system. The magnetic ions \( \text{Cr}^{3+} \) occupy the B site of the spinel structure and there are competing interactions between first-nearest (ferromagnetic) and higher order (antiferromagnetic) neighbours [12]. The samples were made of powder embedded with silicon grease, and packed into the sample holder. The density factor for the \( \text{CsNiFeF}_6 \) sample was 0.60 and 0.46 for the \( \text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4 \) sample. In figures 7 and 8 we have reported the rms value of the noise spectrum (in \( \phi_0/\sqrt{\text{Hz}} \) at the SQUID input) as a function of frequency, for \( \text{CsNiFeF}_6 \) and \( \text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4 \) respectively. The noise spectrum of the empty system is also displayed in the figures. Each spectrum corresponds

Fig. 7. — R.M.S. noise of \( \text{CsNiFeF}_6 \) (upper curve) after subtracting the noise of the empty system (lower curve). The noise is given as the R.M.S. flux at the input of the SQUID. The waiting time after cooling the system is greater than 60 h.

Fig. 8. — The same as figure 7 for \( \text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4 \).
to an average of 256 successive recordings. To avoid ageing effects, which have been reported in previous works [5, 6], recordings have been taken after having spent 70 hours at 4.2 K. A clear 1/f like dependence of the noise power is seen in both materials.

It should be noted that in the present results, due to the improvement on the SQUID characteristics, and on the number of averaged recordings, the resolution of the spectra is strongly improved as compared to the results reported in references [5] and [6]. This allows us to analyse in more details the behaviour of the out of phase susceptibility $\chi''$ which can be calculated from equations (15) and (25) according to the fluctuation dissipation theorem. The result is given in figures 9 and 10, showing a slight departure of the noise power from the 1/f slope. The noise power is well accounted for in both materials by a $1/f^{1+x}$ law, with $x = 0.02$ for $\text{CsNiFeF}_6$ and $x = 0.02$ for $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$. In figure 9 the values of $\chi''$, measured in a conventional susceptibility experiment, have also been reported. Within the experimental errors on the noise calibration (mainly due to the uncertainties on the self inductance values), a satisfactory agreement is obtained between both determinations of the imaginary susceptibility.

It thus appears that the fluctuation dissipation theorem is obeyed in spin glasses below the transition temperature as yet emphasized by several authors [5, 6, 8]; although it is strongly suspected that spin glasses could be non-ergodic. For a non-ergodic medium, the « equilibrium » stationary noise reveals only the existence of quasi equilibrium dynamics in the range of experimental times (here, some multiple of the inverse of the lowest measured frequency), given the material has spent enough time at the measuring temperature (the so called « age » [13]). The good agreement between susceptibility and noise measurements, obtained on the basis of the FDT only shows that similar conditions of quasi equilibrium hold in both cases of experiments. More precisely, for the FDT to be valid, the system must be ergodic, i.e. time averages must be equivalent to thermodynamic averages. Thus a « quasi equilibrium state » could be understood as the average over that part of the phase space the system can visit during the experimental time $t$. As long as the same restrictive conditions apply to both kinds of measurements, a FDT relation could hold between $M^2(t)$ and $\chi''(\omega)$. Moreover, it is worthy to remember that in the present work, powdered samples of about 1 cm$^3$ are used. This results in an effective average over the disorder, which can occult at least to some extend a non-ergodic behaviour. These open problems will be examined in more detail with the aid of a new experimental device allowing variable temperature measurements, as well as coupling to samples of reduced size.

Acknowledgments.

The authors are indebted to H. Bouchiat and P. Monod for their fruitful discussions and remarks. We also thank J. Hammann, M. Alba, E. Vincent and R. Gerard Deneuville.

Appendix A.

The purpose of this appendix is to derive tractable relations giving the filling factor $q$ in real experimental geometries.

The filling factor for a rod-shaped sample coupled to a thin ring surrounding it, has been numerically computed. The notations are the following: $a$ is the radius of the ring, $e$ is the distance between the ring and the edge of the cylinder, and $\lambda$ is the half height of the cylinder (see Fig. 1).
The function $h(r)$ is given by classical magnetostatics [14]; in cylindrical coordinates:

$$
h_p = \frac{kz}{\rho \sqrt{\rho^2 + z^2}} \left[ -K(k) + \frac{a^2 + \rho^2 + z^2}{(a - \rho)^2 + z^2} E(k) \right]
$$

$$
h_z = \frac{k}{\sqrt{\rho^2 + z^2}} \left[ K(k) + \frac{a^2 - \rho^2 - z^2}{(a - \rho)^2 + z^2} E(k) \right]
$$

where

$$
k = \frac{4 a p}{(a^2 + \rho^2 + z^2)}
$$

and $K(k)$ and $E(k)$ are the complete elliptic integrals of first and second kind [15].

The filling factors are given by relations (9) and (22) respectively for short and long range correlations. The problem is to compute the integrals

$$
\int \sum_{\mu} h_\mu^2(r) \, dr \quad \text{and} \quad \frac{1}{V} \sum_{\mu} \left[ \int h_\mu(r) \, dr \right]^2.
$$

For this purpose, reduced quantities are used

$$
\nu = \frac{\rho}{a}, \quad \xi = \frac{z}{a}, \quad \lambda = \frac{\Lambda}{a}
$$

and

$$
\tilde{h}_\mu = ah_\mu.
$$

Then

$$
\int \sum_{\mu} h_\mu^2(r) \, dr = 4 \pi a \int_0^1 \int_0^\lambda \nu (\tilde{h}_\mu^2 + \tilde{h}_z^2) \times \times d\xi \, d\nu = aF_\lambda(\lambda).
$$

In the case of long range correlations $\tilde{h}_z$ is the only non-vanishing term in the integral

$$
\frac{1}{V} \sum_{\mu} \left[ \int h_\mu(r) \, dr \right]^2 = \frac{8 \pi a}{\lambda} \left[ \int_0^1 \int_0^{1-\nu} \nu \tilde{h}_z \, d\xi \, d\nu \right]^2 = aF_1(\lambda)
$$

(A.4)

$\sqrt{F_s(\lambda)}$ and $\sqrt{F_1(\lambda)}$ have the dimension of a permeability (no dimension in e.m.u. units).

Finally, the filling factor reads:

$$
q_i = a \frac{F_i(\lambda)}{4 \pi L_0} \quad i = s \text{ or } 1
$$

where $L_0$ is the self inductance of the ring.

The computed values of $F_s(\lambda)$ and $F_1(\lambda)$ are reported in figures 3 and 4 for different values of $a/e$, and figure 5 represents the variation of $F_s$ and $F_1$ as a function of $a/e$ for $\lambda = 2$. See text for discussion of the results.

Actually, the problem is the calculation of the filling factor for a real winding and not for an ideal thin ring. However, a perturbative calculation has shown that the correction of the previous results do not exceed 10%. This is masked by the imprecision in the determination of the experimental self inductance.

References

[10] Comparison with the results of references [5] and [6] shows that the sensitivity of the SQUID has been strongly improved. Particularly, the effect of $^4$He bath instabilities at low frequencies has been reduced by an order of magnitude.