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Steady state temperature distribution in a cylindrical electrical conductor-Non-linear effects

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Résumé. — Nous résolvons l'équation non linéaire de conduction de la chaleur dans un conducteur cylindrique en régime permanent. Nous supposons que la résistivité électrique ρ et la conductivité thermique λ sont fonctions linéaires de la température. Nous utilisons une méthode approximative de linéarisation optimale et comparons la solution analytique de cette méthode avec la solution numérique de l'équation non linéaire originale.

Abstract. — We solve the non-linear steady state heat conduction equation in a cylindrical conductor. We assume that the electrical resistivity ρ and the thermal conductivity λ are both linear functions of the temperature. We use an approximate optimal linearization method and compare the analytical results of this method with the numerical solution of the original non-linear equation.

1. Introduction.

The resolution of the heat conduction equation using approximate methods has received particular attention in recent years. This is partially due to the development in the experimental methods of measuring parameters such as the electrical resistivity ρ and the thermal conductivity λ as functions of the temperature T [1, 2]. Knowing the exact variation of ρ and λ with temperature, one can determine the steady state temperature distribution by solving the following equation :

$$\nabla \cdot [\lambda(T) \nabla T] = -q(r, T) \quad (1)$$

where $q(r, T)$ is the rate of internal heat production and r is the space parameter. This equation has been solved by various authors [3-6] using approximate methods in the case of rectangular coordinates. However, there are several cases of practical interest where one has to solve equation 1 in cylindrical coordinates which is the problem we wish to present in this paper. One can apply the results of these calculations to the problem of heat transfer in electrical equipment using direct current.

Here we solve equation 1 by assuming the heat production is due to the direct current that flows in a cylindrical conductor. The time dependent case will be considered in another paper. This problem has been solved by Hoffer [7] in the case where ρ is a linear function of temperature but λ is constant. Here, we

assume that both ρ and λ are linear functions of the temperature T . We choose these forms because they are reasonable in the range of temperature which we consider here, although other forms can be accommodated in our theoretical approach.

2. Resolution of the steady state heat equation.

In the case of an infinite, isotropic, cylindrical conductor, equation 1 becomes

$$\lambda(T) \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right] + \frac{d\lambda(T)}{dT} \left(\frac{dT}{dr} \right)^2 + \rho(T) j^2 = 0 \quad (2)$$

where T is the difference of temperature expressed in [K] and corresponds to the increase above the ambient temperature T_0 , r is the radial coordinate, and $j = I/\pi r_0^2$ is the constant direct current density [A/m²]. To solve equation 2 we use the following boundary conditions :

$$\left. \frac{dT}{dr} \right|_{r=0} = 0 \quad (3a)$$

$$\lambda(T) \left. \frac{dT}{dr} \right|_{r=r_0} = -\varepsilon T(r_0) \quad (3b)$$

where r_0 is the radius of the conductor and ε the convective heat transfer coefficient [W K⁻¹ m⁻²]. In equations 2 and 3 $\rho(T)$ and $\lambda(T)$ can be arbitrary functions

of temperature. In the case where λ is constant ($\lambda = \lambda_0$) and ρ is a linear function of temperature

$$\rho(T) = \rho_0(1 + \alpha T). \quad (4)$$

Equation 2 becomes

$$\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} + \alpha c_0 T + c_0 = 0 \quad (5)$$

where $c_0 = \rho_0 j^2 / \lambda_0 [\text{K m}^{-2}]$, ρ_0 and λ_0 are the electrical resistivity and the thermal conductivity at T_0 respectively, and α is a numerical coefficient. In the latter case the boundary condition 3b becomes simply

$$\left. \frac{dT}{dr} \right|_{r=r_0} = -\frac{\varepsilon}{\lambda_0} T(r_0).$$

The solution of equation 5 satisfying these boundary conditions can be obtained as [7]

$$T(r) = \frac{\varepsilon}{\alpha} \frac{J_0(r\sqrt{\alpha c_0})}{\varepsilon J_0(r_0\sqrt{\alpha c_0}) - \lambda_0\sqrt{\alpha c_0} J_1(r_0\sqrt{\alpha c_0})} - \frac{1}{\alpha} [\text{K}] \quad (6)$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions of the first kind [8]. This result has also been obtained by Hoffer [7] by solving the transient case and taking the infinite time limit. It is important to note that for this result to be physically acceptable, one has to make sure that the denominator $\varepsilon J_0(r_0\sqrt{\alpha c_0}) - \lambda_0\sqrt{\alpha c_0} J_1(r_0\sqrt{\alpha c_0})$ remains always positive definite, which introduces a constraint in the choice of the parameters involved in this expression. If one assumes

that both ρ and λ are linear functions of the temperature, which means that in addition to (4), we have :

$$\lambda(T) = \lambda_0(1 + \alpha' T) \quad (7)$$

where α' is a numerical factor. Equation 2 becomes

$$(1 + \alpha' T) \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right] + \alpha' \left(\frac{dT}{dr} \right)^2 + \alpha c_0 T + c_0 = 0. \quad (8)$$

This is a non-linear differential equation which does not have an exact analytical solution. To solve this equation we propose to use the method of optimal linearization [4] which consists of approximating equation 8 by a linear equation of the form

$$\lambda_{\text{op}} \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right] + \rho(T) j^2 = 0 \quad (9)$$

where we have introduced a constant optimal thermal conductivity λ_{op} which will be determined by the procedure described below.

In the present method one replaces λ_{op} by a parameter k and introduces the quantity $I(k)$

$$I(k) = \int_0^{r_0} dr \Delta^2, \quad (10)$$

where Δ is obtained by taking the difference between equations 9 and 2 :

$$\Delta = [k - \lambda(T)] \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right] - \frac{d\lambda(T)}{dT} \left(\frac{dT}{dr} \right)^2. \quad (11)$$

Combining equations 7, 10 and 11 one can put $I(k)$ in the form

$$I(k) = Ak^2 - 2(B + C)k + D, \quad (12)$$

where the coefficients A , B , C and D are given by

$$\begin{aligned} A &= \int_0^{r_0} dr \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right]^2 \\ B &= \int_0^{r_0} dr \lambda(T) \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right] \\ C &= \int_0^{r_0} dr \frac{d\lambda(T)}{dT} \left(\frac{dT}{dr} \right)^2 \left[\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right] \\ D &= \int_0^{r_0} dr \left[\left(\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right) \lambda(T) + \left(\frac{dT}{dr} \right)^2 \frac{d\lambda(T)}{dT} \right]^2 \end{aligned} \quad (13)$$

λ_{op} is obtained by letting

$$\frac{dI(k)}{dk} = 0 \quad (14)$$

whose solution is

$$k = \lambda_{\text{op}} = \frac{B + C}{A}. \quad (15)$$

To calculate λ_{op} explicitly one needs an expression for $T(r)$ which is given by equation 6. This yields :

$$\lambda_{\text{op}} = \lambda_0(1 + \alpha' E) \quad (16a)$$

where E is given by

$$E = F \frac{\int_0^{r_0} dr [J_0^2(r\sqrt{\alpha c_0}) - J_1^2(r\sqrt{\alpha c_0})] J_0(r\sqrt{\alpha c_0})}{\int_0^{r_0} dr J_0^2(r\sqrt{\alpha c_0})} \quad (16b)$$

and

$$F = \frac{\varepsilon}{\alpha [\varepsilon J_0(r_0\sqrt{\alpha c_0}) - \lambda_0\sqrt{\alpha c_0} J_1(r_0\sqrt{\alpha c_0})]}. \quad (16c)$$

Therefore, within the method of optimal linearization, where both ρ and λ are linear functions of T , the temperature distribution is approximately obtained from the linear equation

$$\frac{d^2 T}{dr^2} + \frac{1}{r} \frac{dT}{dr} + \frac{\rho(T)}{\lambda_{\text{op}}} j^2 = 0 \quad (17)$$

with the new boundary condition

$$\left. \frac{dT}{dr} \right|_{r=r_0} = -\frac{\varepsilon}{\lambda_{\text{op}}} T(r_0). \quad (18)$$

The solution of equation 17 satisfying the boundary conditions 3a and 18 is

$$T(r) = \frac{\varepsilon}{\alpha} \frac{J_0(r\sqrt{\alpha c'_0})}{\varepsilon J_0(r_0\sqrt{\alpha c'_0}) - \lambda_{\text{op}}\sqrt{\alpha c'_0} J_1(r_0\sqrt{\alpha c'_0})} - \frac{1}{\alpha} [\text{K}] \quad (19)$$

where $c'_0 = \frac{c_0}{1 + \alpha' E}.$

3. Results.

In figure 1, we have plotted curve 1 corresponding to the temperature distribution $T(r)$ as given by equation 19 using [7]

$$\begin{aligned} r_0 &= 5 \times 10^{-3} [\text{m}], \\ I &= 200 [\text{A}], \\ \rho_0 &= 2.9 \times 10^{-8} [\Omega \text{ m}], \\ \lambda_0 &= 180 [\text{W m}^{-1} \text{ K}^{-1}], \quad \text{Al at 293 [K]} \\ \alpha &= 43 \times 10^{-4} [\text{K}^{-1}] \\ \alpha' &= -3.4 \times 10^{-4} [\text{K}^{-1}] \\ \varepsilon &= 16.6 [\text{W m}^{-2} \text{ K}^{-1}]. \end{aligned} \quad (20)$$

Obviously the value of $\alpha' E$ obtained by using these parameters is so small (-0.01096) that one does not expect to find any significant deviation from the linear case where $\lambda = \lambda_0$. We have also plotted in the same figure curves 2 and 3 corresponding to $\alpha' E = +0.5$ and -0.5 , respectively. These numbers may not correspond to real cases and are chosen in order to enhance artificially the effect of non linearity. We have also solved numerically the non-linear equation 2 by applying a Runge-Kutta method of the 4th order and using HP 9825B calculator.

Fig. 1. — The variation of the temperature as a function of r as given by (8) and (9), using $I = 200$ [A] and $\varepsilon = 16.6$ [$\text{W m}^{-2} \text{K}^{-1}$]: curve 1 : $\alpha' = -3.4 \times 10^{-4}$ [K^{-1}]; curve 2 : $\alpha' E = +0.5$; curve 3 : $\alpha' E = -0.5$.

For the sake of comparison with the approximate solution given by equation 19 we have used the parameters displayed by (20) except for α' which is taken to be -3.4×10^{-4} [K^{-1}] for curve 1, and $\pm 0.5/E$ for curves 2 and 3, respectively. We obtain a good agreement between the exact numerical solution and the approximate analytical solution, to within an accuracy of 10^{-4} [K].

The degree of accuracy of our optimal linearization method is a function of the temperature level within the conductor, or equivalently on the choice of the parameters such as I and ε . To show this dependence, we have plotted the curves in figure 2 choosing the same parameters as in figure 1, except for I and ε which are taken to be $I = 2000$ [A] and $\varepsilon = 300$ [$\text{W m}^{-2} \text{K}^{-1}$].

Fig. 2. — The variation of the temperature as a function of r as given by (8) and (9), using $I = 2000$ [A] and $\varepsilon = 300$ [$\text{W m}^{-2} \text{K}^{-1}$]: curve 1 : $\alpha' = -3.4 \times 10^{-4}$ [K^{-1}]; curve 2 : $\alpha' E = +0.5$; curve 3 : $\alpha' E = -0.5$ (Eq. 8); curve 4 : $\alpha' E = -0.5$ (Eq. 9).

This shows that the optimal linearization method which we have presented here can be used to solve the non-linear steady state heat conduction equation with a fairly high degree of accuracy.

It can be particularly useful when λ depends strongly on temperature, i.e. when α' is large. However, this method is not unique to the case of electrical conductors but can be generalized to other heat conducting systems.

Furthermore, one observes that the temperature is practically constant within the conductor, but it depends strongly on other parameters such as the electrical current I and the heat exchange parameter ε . For the sake of simplicity, we investigate the effects of these parameters on the temperature at the centre $T_m = T(r = 0)$, which is given by

$$T_m = \frac{\varepsilon}{\alpha} \frac{1}{\varepsilon J_0(r_0 \sqrt{\alpha c'_0}) - \lambda_{op} \sqrt{\alpha c'_0} J_1(r_0 \sqrt{\alpha c'_0})} - \frac{1}{\alpha} [\text{K}]. \quad (21)$$

Table I.1. — The coefficient ε characterizing the rate of heat exchange using various stationary fluids [9].

Fluid	Air (Conductor with polished surface)	Air (Conductor with black surface)	Oil	Water	Boiling water
ε [$\text{W m}^{-2} \text{K}^{-1}$]	8	20	300	2 300	4 000

Table I.2. — The coefficient ε characterizing the rate of heat exchange in electrical equipment using air-flow at various speeds [9].

Speed [ms^{-1}]	0	1	5	10	20	50
ε [$\text{W m}^{-2} \text{K}^{-1}$]	20	25	44	70	110	150

Fig. 3. — The variation of T_m as a function of the current I in the case where $\lambda = \lambda_0$: curve 1: $\varepsilon = 16.6$ [$\text{W m}^{-2} \text{K}^{-1}$]; curve 2: $\varepsilon = 300$ [$\text{W m}^{-2} \text{K}^{-1}$]. The dotted line corresponds to the case where λ is a linear function of the temperature with $\alpha' E = -0.5$ and $\varepsilon = 300$ [$\text{W m}^{-2} \text{K}^{-1}$].

In figure 3, we have plotted T_m as a function of the current I using the following parameters [7],

$$\begin{aligned}\rho_0 &= 2.9 \times 10^{-8} \text{ } [\Omega \text{ m}] \\ \alpha &= 43 \times 10^{-4} \text{ } [\text{K}^{-1}] \\ \lambda_0 &= 180 \text{ } [\text{W m}^{-1} \text{K}^{-1}].\end{aligned}$$

Curve 1 corresponds to $\varepsilon = 16.6$ [$\text{W m}^{-2} \text{K}^{-1}$] and has been taken from reference 7. Curve 2 is plotted for $\varepsilon = 300$ [$\text{W m}^{-2} \text{K}^{-1}$] (see tables I).

One observes that ε affects strongly the temperature in the conductor.

The effect of ε on T_m when λ is a linear function of temperature is not significantly different from the limit when $\lambda = \lambda_0$, even by letting $\alpha' E = -0.5$ (see dotted curve in Fig. 3).

4. Conclusions.

In this paper, we have applied the method of optimal linearization to solve the steady state non-linear heat conduction equation for a cylindrical conductor. The analytical results of the linear « optimal » equation have been compared with the numerical resolution of the exact non-linear equation, and the agreement was found to be good. Here, we have chosen the particular case where the parameters $\rho(T)$ and $\lambda(T)$ vary linearly with temperature, but the method of optimal linearization is not restricted to these types of variations only.

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