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A mechanical approach of phase conjugation

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Résumé. — On étudie la conjugaison de phase dans le cas d'un oscillateur mécanique anharmonique : ce dernier est soumis à un couple sinusoidal intense auquel on superpose un couple faible. Dans la réponse supplémentaire liée à ce couple faible on met en évidence une partie conjuguée en phase en relation avec la réponse non linéaire de l'oscillateur ; une analogie rapide est faite avec le rôle en Optique non linéaire du coefficient de susceptibilité non linéaire d'ordre trois.

Abstract. — A mechanical oscillator is used to study phase conjugation : the phase conjugate part of the response of an anharmonic oscillator excited simultaneously by an intense torque and a weak probe torque is calculated. The connections with third order Non Linear Optics susceptibility tensor is presented.

1. Introduction. — If we study the response of a nonlinear mechanical system — such as an anharmonic pendulum — excited simultaneously by an intense and a weak torque having the same frequency \( \omega \) and different phases, we can prove that a part of the response is phase conjugate with the weak excitation.

Generally speaking, phase conjugation can be described in the following scheme : suppose a wave \( A_1 = R[\psi(r) e^{i(\omega t - kr)}] \); then \( A_2 = R[\psi^*(r) e^{i(wt+kr)}] \) is called phase conjugate of \( A_1 \) : to obtain \( A_2 \) from \( A_1 \), we only take the complex conjugate of the spatial terms, without changing the temporal term ; it is equivalent to take the complex conjugate of \( e^{iwt} \), without changing the spatial terms [1]. So, phase conjugation is just as time reversal, and is a very important field in Physics.

In the last ten years, phase conjugation (PC) was intensively studied in Non Linear Optics where it leads to important applications such as compensation of wave front distortions during transmission or amplification through inhomogeneous mediums, and real time holography. The first experiment was carried out by Stepanov et al. [2] in 1970, who produced the conjugate wavefront generation via four waves mixing in a saturable dye solution, and explained their results on the basis of a holographic analogy. Soon afterwards, other possibilities to obtain PC, such as three waves mixing [3] and backward stimulated Brillouin [4] or Raman [5] scatterings were suggested and demonstrated.

In all these phenomena, PC can be described using a simplified description of the third order non linear polarization \( \mathbf{P}_{NL} \) of the medium excited by an optical field \( \mathbf{E}_s \) :

\[
\mathbf{P}_{NL} = \chi^{(3)}(\mathbf{E}_s) \, .
\]  

(1)

If the exciting field \( \mathbf{E}_s \) is the sum of an intense field \( \mathbf{E} = A \sin \omega t \) and a weak field \( \mathbf{e} = a \sin (\omega t + \phi) \) at the same frequency, the Fourier's analysis of \( \mathbf{E}_s \) gives :

\[
\mathbf{E}_s^1 = \frac{3}{4} A^2 a \sin (\omega t - \phi) + \text{other terms} \, .
\]

(2)

The first term of the right hand of (2) is phase conjugate with the probe field \( \mathbf{e} \), the other terms are fundamental or harmonic terms which do not contribute to PC. So \( \mathbf{P}_{NL} \) has a part \( \mathbf{P}_{NL}(\omega + \omega - \omega) \) which is phase conjugate with the probe field :

\[
\mathbf{P}_{NL}(\omega + \omega - \omega) = \frac{3}{4} \chi^{(3)} A^2 a \sin (\omega t - \phi) \, .
\]

(3)

Our purpose in this work is to show that the idea of PC can be extended to mechanical oscillators and, by this way, to a lot of wave processes in which the oscillator can be considered as a source. We have obtained general results ; some of them can be compared to those obtained in formula (3) from optical wave methods.

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2. Non linear vibration of a pendulum. — As a first step, we study a particular mechanical system, the anharmonic pendulum, excited simultaneously by a torque of large amplitude $M$ and a torque of small amplitude $m$ at the same frequency $\omega_e$. We shall calculate the difference $x$ between the responses $X$ and $X'$ of the pendulum to the excitations $M_e (M_e = M + m)$ and $M$, and we shall prove that a part of $x$ is phase conjugate with $m$.

2.1 Equation of Motion of the Pendulum. — The equation of the angular coordinate $x_\theta$ of the pendulum (moment of inertia $I$, natural frequency $\omega_0$, quality factor $Q$) excited by a strong periodic torque $A' \sin \omega_e t'$ and a small perturbation $a' \sin (\omega_e t' + \phi)$ is:

$$\frac{d^2}{dt^2} (x_\theta) + \frac{1}{Q} \frac{d}{dt} (x_\theta) + x_\theta = A' \sin \omega_e t' + a' \sin (\omega_e t' + \phi).$$

It is a reduced equation, where $t = \omega_0 t'$, $\omega = \omega_0/\omega_e$, $a = a'/I\omega^2_0, A = A'/I\omega^2_0$, are dimensionless. $X$ obeys a similar equation:

$$\ddot{X} + \frac{1}{Q} \dot{X} + X = A \sin \omega t.$$  \hspace{1cm} (4)

For low values of $m (x \ll 1)$, $\sin X$ is practically equal to $\sin X + x \cos X$ and the difference between (4) and (5) gives:

$$\ddot{x} + \frac{1}{Q} \dot{x} + x \cos X = a \sin (\omega t + \phi).$$  \hspace{1cm} (6)

2.2 Numerical Solutions of the Equations of Motion. — The solutions of (5) and (6) are computed. The time is divided into small and equal intervals $d$ ($t_n = nd$); the values $X_0, X_1, X_2, ...$, $x_0, x_1, x_2, ...$ corresponding to $t_0, t_1, t_2, ...$ are calculated step by step, if the initial conditions $(X_0, x_0, x_1)$ are given, using the relations (valid when $d$ is very small):

$$\dot{x}_n = \frac{x_{n+1} - x_{n-1}}{2d}, \quad \ddot{x}_n = \frac{x_{n+1} - 2x_n + x_{n-1}}{d^2}$$  \hspace{1cm} (7)

and the similar relations for $X$. Using (7) in (5) and (6), $x_{n+1}$ or $X_{n+1}$ can be deduced from $x_{n-1}$ and $x_n$ or from $X_{n-1}$ and $X_n$. The time interval $d$ is determined by the number $Z$ of intervals in one period of the external torque ($d = 2 \pi/\omega Z$). $Z$ is chosen sufficiently large, so (7) is correct. If we are concerned with the permanent solution, the knowledge of the values $X_n, X_{n+1}, ..., x_n, x_{n+1}, ...$, with $N \geq 4 QZ$ is sufficient (for $N \geq 4 QZ$, the transient response has vanished). The period $T$ of the computed values of $x$ or $X$ is $2 \pi/\omega$, and the fundamental term of $x$, $(x_\theta)_0$, may be written:

$$x_\theta = s \sin \omega t + c \cos \omega t$$  \hspace{1cm} (8)

where $s$ and $c$ are computed in the usual manner:

$$s = \frac{2}{T} \int_{-T/2}^{T/2} x \sin \omega t \, dt \quad c = \frac{2}{T} \int_{-T/2}^{T/2} x \cos \omega t \, dt.$$  \hspace{1cm} (9)

These integrals are calculated using the Simpson's method.

Similar calculations can be made for $X$. $x_\theta$ is related to $\phi$, and is computed for $\phi = 0$ and $\phi = \frac{\pi}{2}$; so the corresponding values of $c$ ($c^0$ and $c^{\pi/2}$) and of $s$ ($s^0$ and $s^{\pi/2}$) are computed.

Relatively to $x$, (6) is linear and the right hand can be written $\cos \phi a \sin \omega t + \sin \phi a \sin (\omega t + \pi/2)$; so $x_\theta$ is given by:

$$x_\theta = x^0 \cos \phi + x^{\pi/2} \sin \phi$$  \hspace{1cm} (10)

$x^0, x^{\pi/2}$ are the values of $x_\theta$ for $\phi = 0, \frac{\pi}{2}$. It is easy to prove that:

$$x_\theta = x_+ + x_-$$  \hspace{1cm} (11)

where:

$$x_- = (x_-)_0 \sin (\omega t - \phi + \alpha_-)$$  \hspace{1cm} (12a)

$$x_+ = (x_+)_0 \sin (\omega t + \phi + \alpha_+)$$  \hspace{1cm} (12b)

Fig. 1. — Amplitude $x_\theta$ of the conjugate part of the response to a probe excitation $a$ versus the amplitude $(X_\theta)_0$ of the fundamental of the response to an intense torque (for various reduced frequency $\omega$ and $Q = 20$).
\( \alpha_- \) is independent of \( \phi \); \( x_- \) is the phase conjugate response. The constants of (12a) are given by:
\[
(x_-)_0^2 = A^2 + B^2 \quad \alpha_- = \arctan(A/B)
\]
where:
\[
A = \frac{1}{2}(s^{\pi/2} + c^{\pi/2}), \quad B = \frac{1}{2}(s^0 - c^{\pi/2}).
\]

We studied numerically \((x_-)_0\) and \(\alpha_-\) for different values of the frequency \(\omega\) and the amplitude \(A\) of the strong torque. The results are displayed in Figure 1; they are given versus the amplitude \((Xf)_0\) of the fundamental of \(X\) and will be examined in the conclusion.

2.3 Analytical Solution of the Equations.
A first order perturbation calculation is given for moderate excitation. For moderate \(X\), we can use:
\[
\sin X = X - X^3/3! \quad \cos X = 1 - X^2/2.
\]
So the equations (5 and 6) become:
\[
\ddot{X} + \frac{1}{Q} \dot{X} + X - A \sin \omega t = X^3/6 \quad (15a)
\]
\[
\ddot{x} + \frac{1}{Q} \dot{x} + x - a \sin (\omega t + \phi) = \frac{1}{2} X^2 x \quad (15b)
\]

The iterated solutions \(X^{(n)}\) and \(x^{(n)}\) of order \(n\) are given by the equations:
\[
\ddot{X}^{(n)} + \frac{1}{Q} \dot{X}^{(n)} + X^{(n)} - A \sin \omega t = (X^{(n-1)})^3/6
\]
\[
\ddot{x}^{(n)} + \frac{1}{Q} \dot{x}^{(n)} + x^{(n)} - a \sin (\omega t + \phi) = x^{(n-1)}(X^{(n-1)})^2/2. \quad (16b)
\]

At zero order the right hands of (16) are nil, and \(X^{(0)}\) is the well known solution of a linear equation:
\[
X^{(0)} = BA \sin (\omega t + \psi) \quad x^{(0)} = Ba \sin (\omega t + \phi + \psi)
\]
where:
\[
1/B^2 = (1 - \omega^2)^2 + (\omega/Q)^2 \\
\psi = \arctan(-\omega/(Q(1 - \omega^2))).
\]

Following (16b), the conjugate part of \(X^{(0)}(X^{(0)})^2/2\),
\[
\left[ \frac{1}{8} B^3 A^2 a \sin (\omega t - \phi + \psi) \right],
\]
gives rise to the phase conjugate part of \(x^{(1)}\) (which is also \(x\) if we take only the first order calculation). So, it is easily found that:
\[
x_- = \frac{1}{8} B^4 A^2 a \sin (\omega t - \phi + 2 \psi). \quad (18)
\]

The result coincides with the calculation obtained previously by use of computer (see Fig. 2) for moderate values of \(A\).

It can be mentioned that some general calculations using the perturbation method are given by Flytzanis et al. [7].

3. Conclusion.
We can draw some inferences from our calculations:
1) If a pendulum is excited simultaneously by a strong periodic torque and a weak one, the fundamental can be separated into three parts:
   a) \(X_f\), the fundamental of the response to the strong excitation,
   b) \(x_+\), the fundamental of the additional response related to the weak excitation, having the same change of phase,
   c) \(x_-\), defined as \(x_+\) but the change of phase of \(x_-\) is minus the change of phase of the weak excitation:
      \(x_-\) is phase conjugate.

The amplitude \((x_-)_0\) of \(x_-\) has been calculated analytically for moderate excitation and computed for moderate and large excitations. It can be noted that the existence of the phase conjugate term is connected to the nonlinearity of the pendulum.

2) When the amplitude \((X_f)_0\) is not too large (for instance when \(\omega = 20\), \(|\omega/\omega_0| < 1\) rad.), the computed value of \((x_-)_0\) coincides with the analytical one obtained by the method of first order pertur-
bation. Therefore, from formula (18), we know the amplitude \((x_-)_0\) of \(x_-\):

\[
(x_-)_0 = \frac{1}{8} B^4 A^2 a = \frac{1}{8} A^2 a[(1 - \omega^2)^2 + (\omega/Q)^2]^{-2}.
\]

(19)

It means that the amplitude of the phase conjugate part of the response of a pendulum to a weak excitation \(a\) can be amplified by the strong one \(A\). The ratio \(\omega\) of the exciting frequency \(\omega_e\) to the natural one \(\omega_0\) has a great influence. If \(\omega\) is far from 1, \((x_-)_0\) decreases quickly (see Fig. 3).

\((x_-)_0\) in (19) can be compared with \(\frac{3}{4} \chi^{(3)} A^2 a\) obtained in optics (3): \([1 - \omega^2]^2 + (\omega/Q)^2]^{-2}\) plays the same role as \(\chi^{(3)}\) but, for high \(A\), this will no longer be true.

3) If \(\omega < 1\), there exists a resonant point which can be interpreted by the fact that a kind of resonance occurs in (5) (see Fig. 1) when the average value of \(\cos X\) is equal to \(\omega^2\). It is one of the results obtained by the use of a mechanical method to describe phase conjugation: there is a value of the exciting torque (or exciting strong field) for which the amplitude of the phase conjugate response has a strong maximum, and we can calculate this value.

The mechanical model is particularly fruitful when the connection with the atomic or molecular model is obvious, such as in the case of phase conjugation by stimulated scatterings. Other kinds of non-linearities, more realistically connected with atomic models, can be studied in the same way.

Fig. 3. — Amplitude \((x_-)_0\) of the conjugate part of the response to a probe excitation \(a\) versus the reduced frequency \(\omega\) \((X)_{h_0} = 1\).

References


