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A simple accurate expression of the potential in electrostatic lenses. Part I : Two cylinder lenses.

P. Bonjour

Institut d'Electronique Fondamentale (*)

Bâtiment 220, Université Paris Sud, 91405 Orsay Cedex, France

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Résumé. — On présente une méthode simple et précise de calcul du potentiel dans les lentilles formées de deux cylindres séparés par un intervalle fini sous la forme d'un développement en fonctions de Bessel. Les propriétés du premier ordre et la constante d'aberration sphérique sont calculées dans un grand domaine de valeurs du rapport des potentiels et de l'intervalle. On compare les résultats obtenus avec ceux donnés par d'autres méthodes.

Abstract. — A simple accurate method is presented of calculating the potential of two-cylinder lenses separated by a finite gap under the form of an expansion of Bessel functions. The first order properties and the spherical aberration constant are calculated over a large range of potential ratios and gap widths. The results are compared with those obtained by various other methods.

1. **Introduction.** — Electrostatic double cylinder lenses being unaffected by the particle ratio e/m are commonly used to focus electron beams as well as to handle ion beams. The optical characteristics of most of these lenses have been described in various papers ; they always result from one of the following methods.

— *Analytical methods.* There is an exact analytical solution of Laplace's equation for two cylinders at different potentials with zero separation between them but no exact solution exists for the case of cylinders separated by a finite gap. Nevertheless, there is an approximate solution in the idealized case when the potential in the gap, along the edge of the two cylinders, is assumed to be varying linearly.

— *Numerical calculation.* The relaxation method is very commonly used for the numerical solution of Laplace's equation or Poisson's equation. The simple five-point formulae are usually chosen, but nine-point ones which improve the precision, though they are more complex to program, have also been used [1]. Relaxation methods have the inherent disadvantage that they give the potential only at nodes on a mesh between the electrodes. The precision increases with the number of points in the mesh, but this number is limited by the core-memory size of the computer and it is difficult to evaluate this precision in a simple way

during the calculation of the iterations. The potential interpolations between the points and the numerical derivations, necessary for the computation of the trajectories and the aberration coefficients, must be done with the greatest care to preserve a fair level of precision.

— *Charge distribution method.* A third method has been recently suggested by Read *et al.* [2]. They consider each cylinder in the lens as a superposition of rings, each of them with a different charge, so that the sum of the potentials created by these rings should have, on each cylinder, a constant value equal to the potential applied to it. The solution obtained in this way is accurate, but to find the charge distribution a set of several hundred simultaneous equations is needed which is solved by inversion of an ill-conditioned matrix.

In this paper, we describe a very accurate method to solve Laplace's equation for the case of cylinders separated by a finite gap. We use the principle of variations to obtain the potential as a series of Bessel functions of orders zero and one respectively. Then the potential can be calculated for any point, and used for the computation of the optical properties of the lenses. The method calls neither for a matrix inversion, nor relaxation on a mesh — both needing a lot of computer memory — and can normally be worked out on a minicomputer.

(*) Laboratoire associé au C.N.R.S.

2. Principle of the method. — In a volume limited by a closed surface, the electrostatic potential obeys Laplace's equation $\nabla^2 V = 0$. By selecting among the solutions of this equation the one which has the values V_B imposed on the surface, the one and only solution of the problem can be found. The electric field being defined as $\mathbf{E} = -\nabla V$, it is known [3] that the integral

$$\varepsilon = \frac{\varepsilon_0}{2} \int (\nabla V)^2 dv$$

is minimal : a slight variation of V from its value determines a second order variation of ε . ε is proportional to the potential energy stored in the system and it has a minimal value. If an approximate solution V' depending on arbitrary parameters $\lambda, \mu, \dots$ is determined for this problem, the principle of variations leads to $\varepsilon(V') \geq \varepsilon(V)$ and the conditions

$$\frac{\partial \varepsilon(V')}{\partial \lambda} = 0, \quad \frac{\partial \varepsilon(V')}{\partial \mu} = 0, \dots$$

give the solution closest to the exact one, among all the various possible solutions.

This method has been used by Cook *et al.* [4] for the case of two cylinders between which the potential at the radius of the cylinders varies linearly across the gap and it led them, of course, to previously known results.

We are using this method for the case when the variation along the edge in the gap is assumed to be a third degree function. This function depends on a parameter a , and so does the potential within the cylinders. The value of parameter a is then determined by writing that the electrostatic energy is minimal.

3. Calculation of the potential. — We consider the two cylinder system shown in figure 1. The potentials of the two cylinders are V_1 and V_2 and their common radius R is chosen as length unit. Since the solution depends only on the V_2/V_1 ratio, it can be assumed without a loss of generality that $V_1 = 0$ and $V_2 = 1$. We use cylindrical coordinates with the origin placed in the centre of the gap.

Fig. 1. — The geometry of a two cylinder lens showing the system of coordinates and the dimensions. Each region I, II and III has its own analytical expression for the potential. The symmetry plane is used as reference plane for the position of the cardinal elements.

Between points A and B, we assume that the potential is expressed as a polynomial

$$V(z) = \frac{a+1}{g} (z + g/2) - \frac{3a}{g^2} (z + g/2)^2 + \frac{2a}{g^3} (z + g/2)^3. \quad (1)$$

This potential has a zero value for $z = -g/2$ and equals 1 for $z = g/2$. a is a parameter which characterizes its degree of cubicity : for $a = 0$, the potential varies linearly between A and B. For $a = 2$, we get a pure cubic

$$V(z) = \left(\frac{2z}{g}\right)^3 + \frac{1}{2}.$$

The curve of $V(z)$ possesses a centre of symmetry at $V = 0.5$ for $z = 0$. The function (1) can be expressed with functions V_n of the form

$$\begin{aligned} V_n(z) &= 0 & z \leq 0 \\ V_n(z) &= z^n & z \geq 0 \end{aligned} \quad (2)$$

in which n is a positive integer.

By carrying (2) into (1)

$$V(z) = \frac{a+1}{g} V_1(z + g/2) - \frac{3a}{g^2} V_2(z + g/2) + \frac{2a}{g^3} V_3(z + g/2).$$

This expression is similar to (1) between A and B. It also represents the potential on the cylinder in region I since it is null everywhere.

(1) can also be written as

$$V(z) = \frac{a+1}{g} (z - g/2) + \frac{3a}{g^2} (z - g/2)^2 + \frac{2a}{g^3} (z - g/2)^3 + 1. \quad (3)$$

This expression has the same centre of symmetry as (1). The difference between (1) and (3) is null. It represents in particular the potential along tube III when 1 is added to it. By introducing the functions $V_n(z - g/2)$ in the difference and adding 1, we finally obtain

$$\begin{aligned} V(z) &= \frac{a+1}{g} (V_1(z + g/2) - V_1(z - g/2)) \\ &\quad - \frac{2a}{g^2} (V_2(z + g/2) + V_2(z - g/2)) \\ &\quad + \frac{2a}{g^3} (V_3(z + g/2) - V_3(z - g/2)). \end{aligned} \quad (4)$$

This expression is null for $z \leq -g/2$, it is similar to (1) for $-g/2 \leq z \leq g/2$ and equals 1 for $z \geq g/2$. There-

Table I. — *The solutions of Laplace's equation when the boundary potential obeys the equations (2), according to Bertram [5].*

$$\begin{array}{ll}
 z < 0 & z > 0 \\
 V_1(r, z) = \sum \frac{J_0(\mu r)}{\mu^2 J_1(\mu)} e^{\mu z} & z + \sum \frac{J_0(\mu r)}{\mu^2 J_1(\mu)} e^{-\mu z} \\
 V_2(r, z) = \sum \frac{J_0(\mu r)}{\mu^3 J_1(\mu)} e^{\mu z} & z^2 + \sum \frac{2 J_0(\mu r)}{\mu^3 J_1(\mu)} (2 - e^{-\mu z}) \\
 V_3(r, z) = \sum \frac{J_0(\mu r)}{\mu^4 J_1(\mu)} e^{\mu z} & z^3 + \sum \frac{6 J_0(\mu r)}{\mu^3 J_1(\mu)} \left(2z + \frac{e^{-\mu z}}{\mu} \right)
 \end{array}$$

fore, it represents the potential in the three regions for $r = 1$.

The functions $V_n(z)$ have been defined by Bertram

[5], who has also given the solution of Laplace's equation $V_n(r, z)$ within an indefinite cylinder when the potential on the limit is $V_n(z)$. Table I summarises his results for n varying from 1 to 3.

$J_0(x)$ and $J_1(x)$ are the Bessel functions of order zero and one respectively, and the summations are taken on all the positive roots μ of the equation $J_0(x) = 0$.

When the potential on the boundary of the system of two cylinders is expressed by (4), the potential inside it is then

$$\begin{aligned}
 V(r, z) = & \frac{a+1}{g} \{ V_1(r, z+g/2) - V_1(r, z-g/2) \} \\
 & - \frac{3a}{g^2} \{ V_2(r, z+g/2) + V_2(r, z-g/2) \} \\
 & + \frac{2a}{g^3} \{ V_3(r, z+g/2) - V_3(r, z-g/2) \}. \quad (5)
 \end{aligned}$$

4. Calculation of the energy and of the parameter a . — Before making this calculation, it must be noticed that the energy of the system whose potential is defined by (5) must have the form

$$\varepsilon = \left(\frac{a+1}{g} \right)^2 W_{11} + \left(\frac{3a}{g^2} \right)^2 W_{22} + \left(\frac{2a}{g^3} \right)^2 W_{33} - \frac{6a(a+1)}{g^3} W_{12} + \frac{4a(a+1)}{g^4} W_{13} - \frac{12a^2}{g^5} W_{23}. \quad (6)$$

The terms W_{ii} come from V_i squared terms, while the W_{ij} ones come from terms of products $V_i V_j$.

ε is a quadratic form of a , whose minimum occurs for the particular value

$$a = - \frac{W_{11} - \frac{3}{g} W_{12} + \frac{2}{g^2} W_{13}}{W_{11} - \frac{6}{g} W_{12} + \frac{4}{g^2} W_{13} + \frac{9}{g^2} W_{22} - \frac{12}{g^3} W_{23} + \frac{4}{g^4} W_{33}}. \quad (7)$$

Because of the antisymmetry of $V(r, z)$, the total energy ε can be obtained through either one of the two integrals

$$\begin{aligned}
 \varepsilon &= 2 \int_0^1 \int_{-\infty}^{-g/2} \left\{ \left(\frac{\partial V_I}{\partial z} \right)^2 + \left(\frac{\partial V_I}{\partial r} \right)^2 \right\} dr dz + 2 \int_0^1 \int_{-g/2}^0 \left\{ \left(\frac{\partial V_{II}}{\partial z} \right)^2 + \left(\frac{\partial V_{II}}{\partial r} \right)^2 \right\} dr dz \\
 \varepsilon &= 2 \int_0^1 \int_0^{g/2} \left\{ \left(\frac{\partial V_{II}}{\partial z} \right)^2 + \left(\frac{\partial V_{II}}{\partial r} \right)^2 \right\} dr dz + 2 \int_0^1 \int_{g/2}^{\infty} \left\{ \left(\frac{\partial V_{III}}{\partial z} \right)^2 + \left(\frac{\partial V_{III}}{\partial r} \right)^2 \right\} dr dz
 \end{aligned}$$

in which V_I , V_{II} , V_{III} are the expressions of the potential in the three regions of figure 1, obtained by carrying the suitable expressions from table I into (5). When we take into account the orthogonality of the Bessel functions in the interval $0 \leq r \leq 1$, we finally obtain the following expressions for W_{ij}

$$\begin{aligned}
 W_{11} &= \frac{g}{2} - 2 \sum \frac{1}{\mu^3} (1 - e^{-\mu g}) \\
 W_{22} &= \frac{2g^3}{3} + 8g \sum \frac{1}{\mu^4} (2 - e^{-\mu g}) + 8 \sum \frac{1}{\mu^5} (1 + e^{-\mu g}) \\
 W_{33} &= \frac{9g^5}{10} + 48g^3 \sum \frac{1}{\mu^4} - 36g^2 \sum \frac{1}{\mu^5} (1 - e^{-\mu g}) + 72g \sum \frac{1}{\mu^6} - 72 \sum \frac{1}{\mu^7} (1 - e^{-\mu g}) \\
 W_{12} &= \frac{g^2}{2} - 2g \sum \frac{1}{\mu^3} (1 - e^{-\mu g}) + 4 \sum \frac{1}{\mu^4} (1 - e^{-\mu g}) \\
 W_{13} &= \frac{g^3}{2} - 3g^2 \sum \frac{1}{\mu^3} (1 - e^{-\mu g}) + 12g \sum \frac{1}{\mu^4} - 12 \sum \frac{1}{\mu^5} (1 - e^{-\mu g}) \\
 W_{23} &= \frac{3g^4}{4} + 24g^2 \sum \frac{1}{\mu^4} + 6g^2 \sum \frac{1}{\mu^4} (1 - e^{-\mu g}) - 12g \sum \frac{1}{\mu^5} (1 - e^{-\mu g}) + 24 \sum \frac{1}{\mu^6} (1 - e^{-\mu g}). \quad (8)
 \end{aligned}$$

For the calculation of a from (8) and (7), as well as that of the potential by means of (5), we have to make the summation of series of terms which contain the roots of the order zero Bessel function and the values of the order one Bessel function for these roots. These values are tabulated in various books [6]. As soon as the order n of the root is above 10 or 15, there are asymptotical expansions [7] to calculate the roots μ_n and $J_1(\mu_n)$ in terms of n .

The calculation of the expressions (8) and of the potential must be carried out with care, as we deal with alternate series in which the terms decrease very slowly, especially when z is close to $\pm g/2$. For instance, it is known that

$$\sum_{n=1}^{n=\infty} \frac{1}{\mu_n J_1(\mu_n)} = 0.5.$$

The sum S_{100} of the 100 first terms is 0.464 639... and the sum S_{101} of the 101 first terms is 0.535 136... The average of these two terms is 0.499 913... : it is a better approximation. The summation of the alternate series is greatly improved by using Shanks's e_1^m process [8]. Starting from three successive sums S_{n-1} , S_n and S_{n+1} , a better approximation T_n of the sum of the series is obtained by using the relation

$$T_n = \frac{S_{n+1} \times S_{n-1} - S_n^2}{S_{n+1} - 2 S_n + S_{n-1}}.$$

For $n = 100$, we obtain here $T_{100} = 0.500\,000\,44...$. The e_1^m process can be applied another time to the T_n terms for another improvement of the summation. For the above series, we then obtain $S = 0.5 \pm 10^{-8}$ when $n = 27$, while $S_{27} = 0.565\,7...$. This is not surprising since all the information concerning the formation of the successive terms in the series is already entirely contained in the first terms.

The expression (6) exactly renders the energy of the system when the potential along $r = 1$ is represented by (5), because the energy stored depends only on the potential on the boundary. The approximation of the method lies in the not rigorously exact representation of the potential between A and B by (1), and a further improvement can be made by replacing (1) by a fifth degree polynomial. Because of the anti-symmetry of $V(z)$, it depends on two parameters, a and b , which have to be determined by the same process.

5. Results : axial potential of two cylinder lenses. —

The calculation of parameter a gives the values shown in figure 2 as a function of g/R . It is to be remembered that a does not depend on the respective potentials V_1 and V_2 of the two cylinders. a has a 1 value for a lens with a zero gap and it decreases slowly as the gap increases.

It is interesting to compare the hypothesis of the cubic variation of the potential in the gap along the

Fig. 2. — The value of parameter a as a function of the gap width.

edge of the cylinders (referred to as the non linear hypothesis in the rest of the text) with those found by other authors.

By derivating (1), it is found that the slope of the potential curve for $z = 0$ and $r = 1$ is

$$m = 1 - a/2$$

the quantity $1/g$ (the gradient of the potential varying linearly in the gap) being taken as unity. For $g = 0.2$, we obtain $m = 0.524$. Figure 2 in [2] shows that Read's method of charge distribution gives $m = 0.65$. The relaxation method used by Natali *et al.* gives a value $m = 0.82$ obtained from table 3 in [1]. It is to be noticed that Read's value is close to ours, while Natali's is further from it and is near the one obtained from the hypothesis of the potential varying linearly in the gap (referred to as linear hypothesis from now on).

In table II, the potentials we have calculated along

Table II. — The axial potential for the lens with $g = 0.2$, $V_1 = 0$ and $V_2 = 1$, showing our results compared with those of various authors.

z	$V(0, z)$ Read	$V(0, z)$ Our results	$V(0, z)$ Linear hypo.	$V(0, z)$ Natali
0	0.500 000	0.500 000	0.500 000	0.500 000
0.1	0.565 297	0.565 345	0.565 497	0.565 468
0.2	0.628 285	0.628 376	0.628 658	0.628 601
0.3	0.686 995	0.687 112	0.687 485	0.687 408
0.5	0.786 562	0.786 691	0.787 120	0.787 031
0.7	0.860 043	0.860 141	0.860 511	0.860 434
1	0.928 897	0.928 921	0.929 149	0.929 103
1.4	0.972 310	0.972 244	0.972 342	0.972 324
2	0.993 456	0.993 393	0.993 417	0.993 414
3	0.999 411	0.999 403	0.999 405	0.999 405
4	0.999 947	0.999 946	0.999 946	0.999 946
5	0.999 995	0.999 995	0.999 995	0.999 995

the axis are compared to the potentials obtained by the other authors. In this example, the potentials are given for positive abscissae and a gap width of 0.2 R.

The second column shows the results of Read's charge distribution method. The third column shows our results and the fourth those obtained from the linear hypothesis ($a = 0$). The fifth column shows the potentials calculated by Natali. Our results agree fairly well with Read's with a difference not over 0.000 1. Those of Natali are very close to those obtained from the linear hypothesis with a difference of about 0.000 1. But between these two groups of results, there are differences which go as far as 0.000 4. The non linear hypothesis and the charge distribution method give similar results which fairly render the properties of the thin-walled lenses, while the relaxation method or the linear hypothesis are better at rendering the properties of thick-walled lenses (thickness equal or superior to 2 R).

Fig. 3. — The axial potential V as a function of z/R for three g values equal to 0.2, 1 and 2, drawn in full lines. The dotted curves show the axial potential in the linear hypothesis.

Figure 3 shows the axial potential for the three lenses $g = 0.2, 1$ and 2 as a function of z/R . The results of the non linear hypothesis are drawn in full lines, while the dotted lines show those of the linear hypothesis. For $g = 0.2$, the two curves cannot be distinguished. In the other two cases, the non linear hypothesis gives a lower potential than the linear one, the maximum difference being obtained for $z \simeq g$.

The axial potential $\mathcal{V}(z)$ of a lens when the cylinder potentials are V_1 and V_2 (the origin of the potentials being the one in which the electrons have a zero velocity) is obtained from the function $V(r, z)$ given from (5) by

$$\mathcal{V}(z) = (V_1 + V_2)/2 + (V_2 - V_1) V(0, z)/2.$$

6. Optical properties of two cylinder lenses. —

The first order properties of the lenses are obtained by integrating the equation of the paraxial rays as modified by Picht [9]

$$R'' + \frac{3}{16} T^2 R = 0 \quad (9)$$

in which the ray equation being $r(z)$, $R = r^{1/4}$ and $T = \mathcal{V}'/\mathcal{V}$. These properties are expressed by means of the cardinal elements : principal planes and focal lengths (Fig. 4). The image of an object placed on

Fig. 4. — The definition of the cardinal elements of a lens. $P1$ is the object principal plane, F_0 the object focus, $F1$ the object focal length and $ZF1$ the object focus position measured from the symmetry plane. $P2$, F_1 , $F2$ and $ZF2$ are the same quantities from the image space.

the revolution axis is affected by the spherical aberration characterized by the coefficient C_s which is given by the following integral, calculated along a particular trajectory

$$C_s = \frac{1}{64 \mathcal{V}_0^{1/2}} \times \int R^4 \mathcal{V}^{1/2} (3 T^4 + 4 T'^2 - 5 T' T^2 - T T'') dz \quad (10)$$

in which T and T' have been defined above, R corresponds to a trajectory starting from the object with a unity slope and the integral is calculated between the object and the image. The equation (9) has been integrated by using the predictor-modificator-corrector method of Hamming [11] which gives an upper limit of the integration error for each step in the integration. The function $T(z)$ has been calculated from (5) and its analytical derivative.

We have calculated the optical parameters of three lenses with the g values 0.2, 1 and 2. They are accelerating lenses ($V_2 > V_1$) with the ratio V_2/V_1 varying according to a geometrical progression with a ratio $(10)^{1/12}$. Table III gives the focal lengths object $F1$ and image $F2$ and the positions of the object focus $ZF1$ and of the image focus $ZF2$. $SP1$ is the integral (10) calculated from $-\infty$ to $+\infty$ for a ray crossing the axis in F_0 with a unity slope : it is the spherical aberration constant for an image located to infinity. When the object moves to infinity, C_s becomes infinitely large, but the magnification M tends to zero and the product $C_s M^4$ has a finite limit value $SP2$ given in the table.

For decelerating lenses ($V_2 < V_1$), table III can be used as follows : the first column gives the V_1/V_2

ratio, the $F1$ and $F2$ values must be reversed and so must $ZF1$ and $ZF2$,

$$(SP1)_{\text{decel}} = (SP2)_{\text{accel}} \times (V_1/V_2)^{3/2},$$

and

$$(SP2)_{\text{decel}} = (SP1)_{\text{accel}} \times (V_1/V_2)^{3/2}.$$

These figures confirm, as it has been already pointed out by Slodzian [12] in the case of unipotential lenses, that accelerating lenses are affected by a smaller

spherical aberration and are better in that way than decelerating lenses.

Table IV enables us to compare some of our results with those obtained by other methods. For $g = 0.2$, we have used the results of Read and Harting [13] and those of Natali *et al.* [1], adding the results given by the linear hypothesis. The value of the object focal length $F1$ is the same, but a slight dispersion appears for the smaller values of the ratio V_2/V_1 . For the spherical aberration constant $SP1$, we have used Read's CSO quantity and Natali's $m_{13} f_2$ quantity.

Table III. — *The values of the cardinal elements and the spherical aberration constants as a function of the voltage ratio V_2/V_1 for lenses having $g = 0.2, 1$ and 2 . A number $x.xxx \pm yy$ should be read as $x.xxx \cdot 10^{\pm yy}$.*

$g = 0.2$

V_2/V_1	$F1$	$F2$	$ZF1$	$ZF2$	$SP1$	$SP2$
1.2	3.549 + 02	3.887 + 02	3.715 + 02	3.714 + 02	2.841 + 07	2.367 + 07
1.5	6.820 + 01	8.353 + 01	7.556 + 01	7.540 + 01	2.248 + 05	1.497 + 05
1.8	3.123 + 01	4.189 + 01	3.628 + 01	3.606 + 01	2.356 + 04	1.304 + 04
2.2	1.668 + 01	2.474 + 01	2.046 + 01	2.016 + 01	3.954 + 03	1.781 + 03
2.7	1.013 + 01	1.664 + 01	1.316 + 01	1.279 + 01	9.764 + 02	3.552 + 02
3.3	6.779 + 00	1.232 + 01	9.352 + 00	8.900 + 00	3.227 + 02	9.488 + 01
3.9	5.086 + 00	1.004 + 01	7.390 + 00	6.875 + 00	1.478 + 02	3.631 + 01
4.7	3.832 + 00	8.308 + 00	5.916 + 00	5.330 + 00	6.932 + 01	1.389 + 01
5.6	3.025 + 00	7.158 + 00	4.954 + 00	4.302 + 00	3.718 + 01	6.144 + 00
6.8	2.390 + 00	6.233 + 00	4.191 + 00	3.465 + 00	2.022 + 01	2.692 + 00
8.2	1.947 + 00	5.576 + 00	3.655 + 00	2.857 + 00	1.201 + 01	1.298 + 00
10.0	1.599 + 00	5.055 + 00	3.231 + 00	2.358 + 00	7.347 + 00	6.364 - 01
12.0	1.355 + 00	4.693 + 00	2.935 + 00	1.992 + 00	4.898 + 00	3.469 - 01
15.0	1.126 + 00	4.361 + 00	2.659 + 00	1.631 + 00	3.141 + 00	1.744 - 01
18.0	9.806 - 01	4.160 + 00	2.485 + 00	1.387 + 00	2.267 + 00	1.037 - 01
22.0	8.522 - 01	3.997 + 00	2.333 + 00	1.159 + 00	1.637 + 00	6.077 - 02
27.0	7.471 - 01	3.882 + 00	2.211 + 00	9.584 - 01	1.212 + 00	3.666 - 02
33.0	6.636 - 01	3.812 + 00	2.117 + 00	7.868 - 01	9.290 - 01	2.316 - 02

$g = 1$

V_2/V_1	$F1$	$F2$	$ZF1$	$ZF2$	$SP1$	$SP2$
1.2	4.035 + 02	4.421 + 02	4.224 + 02	4.223 + 02	2.964 + 07	2.470 + 07
1.5	7.749 + 01	9.491 + 01	8.585 + 01	8.567 + 01	2.352 + 05	1.568 + 05
1.8	3.544 + 01	4.755 + 01	4.118 + 01	4.091 + 01	2.473 + 04	1.373 + 04
2.2	1.890 + 01	2.803 + 01	2.319 + 01	2.283 + 01	4.169 + 03	1.892 + 03
2.7	1.145 + 01	1.881 + 01	1.489 + 01	1.444 + 01	1.034 + 03	3.819 + 02
3.3	7.645 + 00	1.389 + 01	1.056 + 01	1.002 + 01	3.432 + 02	1.035 + 02
3.9	5.722 + 00	1.130 + 01	8.330 + 00	7.716 + 00	1.577 + 02	4.014 + 01
4.7	4.299 + 00	9.320 + 00	6.655 + 00	5.958 + 00	7.418 + 01	1.562 + 01
5.6	3.383 + 00	8.005 + 00	5.563 + 00	4.787 + 00	3.988 + 01	7.021 + 00
6.8	2.664 + 00	6.946 + 00	4.696 + 00	3.834 + 00	2.172 + 01	3.135 + 00
8.2	2.163 + 00	6.192 + 00	4.087 + 00	3.142 + 00	1.291 + 01	1.538 + 00
10.0	1.769 + 00	5.593 + 00	3.606 + 00	2.574 + 00	7.893 + 00	7.679 - 01
12.0	1.493 + 00	5.173 + 00	3.270 + 00	2.158 + 00	5.257 + 00	4.248 - 01
15.0	1.235 + 00	4.785 + 00	2.956 + 00	1.747 + 00	3.364 + 00	2.170 - 01
18.0	1.072 + 00	4.547 + 00	2.758 + 00	1.470 + 00	2.421 + 00	1.303 - 01
22.0	9.276 - 01	4.351 + 00	2.586 + 00	1.211 + 00	1.742 + 00	7.703 - 02
27.0	8.097 - 01	4.207 + 00	2.448 + 00	9.848 - 01	1.284 + 00	4.669 - 02
33.0	7.162 - 01	4.114 + 00	2.341 + 00	7.910 - 01	9.792 - 01	2.955 - 02

$g = 2$

V_2/V_1	$F1$	$F2$	$ZF1$	$ZF2$	$SP1$	$SP2$
1.2	5.179 + 02	5.674 + 02	5.421 + 02	5.420 + 02	3.565 + 07	2.971 + 07
1.5	9.929 + 01	1.216 + 02	1.100 + 02	1.097 + 02	2.851 + 05	1.906 + 05
1.8	4.530 + 01	6.077 + 01	5.265 + 01	5.227 + 01	3.024 + 04	1.694 + 04
2.2	2.407 + 01	3.570 + 01	2.956 + 01	2.905 + 01	5.152 + 03	2.388 + 03
2.7	1.452 + 01	2.385 + 01	1.891 + 01	1.827 + 01	1.292 + 03	4.958 + 02
3.3	9.640 + 00	1.751 + 01	1.336 + 01	1.259 + 01	4.327 + 02	1.386 + 02
3.9	7.179 + 00	1.418 + 01	1.050 + 01	9.628 + 00	2.001 + 02	5.525 + 01
4.7	5.361 + 00	1.162 + 01	8.354 + 00	7.368 + 00	9.454 + 01	2.216 + 01
5.6	4.193 + 00	9.921 + 00	6.956 + 00	5.864 + 00	5.093 + 01	1.024 + 01
6.8	3.277 + 00	8.546 + 00	5.847 + 00	4.640 + 00	2.774 + 01	4.700 + 00
8.2	2.641 + 00	7.563 + 00	5.069 + 00	3.751 + 00	1.645 + 01	2.361 + 00
10.0	2.143 + 00	6.776 + 00	4.455 + 00	3.023 + 00	1.001 + 01	1.203 + 00
12.0	1.796 + 00	6.220 + 00	4.027 + 00	2.490 + 00	6.630 + 00	6.749 - 01
15.0	1.472 + 00	5.700 + 00	3.627 + 00	1.965 + 00	4.204 + 00	3.489 - 01
18.0	1.267 + 00	5.376 + 00	3.375 + 00	1.612 + 00	3.000 + 00	2.104 - 01
22.0	1.087 + 00	5.100 + 00	3.156 + 00	1.284 + 00	2.135 + 00	1.245 - 01
27.0	9.410 - 01	4.890 + 00	2.980 + 00	9.980 - 01	1.556 + 00	7.516 - 02
33.0	8.254 - 01	4.741 + 00	2.844 + 00	7.549 - 01	1.172 + 00	4.717 - 02

Table IV. — A comparison of our results with those of Natali [1] and Read [13] for lenses having $g = 0.2$ and 2. $g = 0.2$

V_2/V_1	$F1$				$SP1$		
	Our results	Linear hypo.	Natali	Read	Our results	Natali	Read
2	21.99	21.95	21.98	22.10	8 648	8 858	8 680
5	3.51	3.51	3.51	3.54	55.14	57.7	55.4
10	1.60	1.60	1.60	1.60	7.35	7.28	7.40
20	0.91	0.91	0.91	0.92	1.90	1.87	1.92

 $g = 2$

V_2/V_1	$F1$				$SP1$		
	Our results	Linear hypo.	Read	Read thick	Our results	Read	Read thick
2	31.81	30.22	33.90	31.48	11 190	12 440	11 160
5	4.90	4.70	5.18	4.90	75.35	84.40	75.20
10	2.14	2.08	2.24	2.16	10.01	11.10	10.22
20	1.17	1.14	1.22	1.20	2.50	2.72	3.18

Our values are about 0.5 % less than those given by Read *et al.* Compared with the results of Natali *et al.*, the difference may be as much as 4 % and it is either positive or negative, depending on the values of the ratio V_2/V_1 .

For $g = 2$, the only published results are those of Read *et al.* They have calculated the potential of lenses made from cylinders under the assumption that the thickness of the cylinder wall is either null or equal to the radius. Our values of $F1$ are less than those given for thin walled-lenses in a range varying from 4 to 6 %.

But for thick-walled lenses, there is generally a good agreement, with some differences of about 1 %. We reach the same conclusions when the spherical aberration is concerned.

We suggest the following explanation : for a thin-walled cylinder, the field dV/dz in points A and B in figure 1 is infinite. In our non linear hypothesis, the field in A or B is equal to $(V_2 - V_1)(a + 1)/g$. This finite value is not suitable for a thin-walled cylinder, but it gives a fair rendering of the real value of the field in A and B when the cylinder is of finite thickness.

7. Conclusion. — We have presented a simple accurate method of calculating the potential inside two cylinder electrostatic lenses; our method is particularly suitable with the wall thickness ranging about the value of the radius.

We obtain an analytical expression in the form of an expansion whose numerical computation is very easy to work out even with a minicomputer. Then, we have calculated the first order properties of these lenses for various values of the gap through a large variation of the potential ratio and the corresponding spherical aberration constants.

This calculation gives accurate results since inter-

polations and numerical derivations, which are always inaccurate, are avoided. Since the analytical expressions of the potential and of its derivatives are known inside the lenses in all points located outside the axis, the integration of the fundamental law of dynamics can be achieved with a numerical method and it can be used to calculate accurately large aperture beams.

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