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ON THE $\cos \varphi$ TERM IN JOSEPHSON'S EXPRESSION FOR THE TUNNELING CURRENT

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Résumé. — L'expression du courant d'une jonction Josephson contient un terme en $\cos \varphi$ auquel on s'est intéressé récemment. Nous présentons dans cet article, le résultat de calculs numériques de ce terme. Bien qu'il affecte théoriquement le courant alternatif d'une jonction polarisée par une tension continue, nous montrons qu'il ne peut pas être mis en évidence expérimentalement par une mesure relative de ce courant. Nous ne considérons que les expériences pour lesquelles on a observé — jusqu'ici — un effet dû à ce terme en $\cos \varphi$. Nous mentionnons aussi quelques exemples de calculs des caractéristiques I - V .

Abstract. — Recently there has been some interest in the $\cos \varphi$ dependent term in the expression for the tunneling current given by B. D. Josephson. In this paper we present the result of numerical calculations on this term and it is shown that although the $\cos \varphi$ term theoretically affects the ac-current in a dc-voltage biased junction the term cannot — in practice — be verified from a relative measurement of this current. In addition we consider the only experiments in which the $\cos \varphi$ term — so far — has been shown to have observable effects. Also some model calculations on I - V characteristics are mentioned.

1. Numerical results for the $\cos \varphi$ term. — From microscopic theory one finds in the case where a tunnel junction is dc-voltage biased that the current through the junction is given by

$$I(t) = \text{Im} \{ I_q(V) + I_p(V) e^{i\varphi(t)} \} \quad (1.1)$$

where $\varphi = 2 |e| V t / \hbar + \varphi_0$ and the two complex functions I_q and I_p are determined by the properties of the two superconductors forming the junction.

This result was obtained first by B. D. Josephson [1], realizing that also pairs contribute to the tunneling current (the second term in eq. (1.1)). Evaluating the imaginary part and introducing the notation of B. D. Josephson [2], [3] which has gradually become standard the current can be written

$$I(t) = \sigma_0(V) V + j_1(V) \sin \varphi + \sigma_1(V) V \cos \varphi. \quad (1.2)$$

For a simple theoretical analysis often only the $\sin \varphi$ term is retained. If losses are important also the σ_0 term is included. Normally both $j_1(V)$ and $\sigma_0(V)$ are replaced by constants. The $\cos \varphi$ term, however, has been neglected until the plasma resonance experiment performed by N. F. Pedersen *et al.* [4], [5] indicated its presence in a real junction.

The three quantities $\sigma_0(V)$, $\sigma_1(V)$, and $j_1(V)$ can be expressed by integrals which for finite temperatures have to be evaluated numerically. At zero temperature they can — as shown by Werthamer [6] — be written in terms of the complete elliptic integrals.

For a tunnel junction made of equal superconductors the amplitude $\sigma_1(V) V$ of the $\cos \varphi$ term is

shown in figure 1 as a function of the voltage and with the reduced temperature as parameter. For voltages above the gap voltage $V_g = 2 \Delta(T) / |e|$, $\sigma_1(V) V$ is negative and for $V \rightarrow \infty$ it tends to zero as $V^{-1} \ln V^{-1}$. The absolute value of the discontinuity at V_g is equal to the discontinuity of the quasiparticle current $\sigma_0(V) V$ at the gap voltage and to the zero voltage value of $j_1(V)$ namely

$$\frac{\pi}{4} \text{th} \left\{ \frac{\Delta(T)}{2 kT} \right\} \sigma_n V_g,$$

where σ_n is the normal state conductivity of the junction. Below the gap voltage $\sigma_1(V) V$ is positive and

FIG. 1. — The quantity $\sigma_1(V) V$ in units of $I_n = \sigma_n 2 \Delta(0) / |e|$ for the BCS value of the gap parameter $\Delta(T)$. For comparison is in dash-dot line shown the normal state current (multiplied by -1).

it has been shown that both $\sigma_1(V)$ and $\sigma_0(V)$ have a logarithmic singularity at zero voltage [7].

In figure 2 is shown the ratio between the $\cos \varphi$ amplitude and the quasiparticle current, i. e., $\sigma_1(V)/\sigma_0(V)$. As seen $|\sigma_1(V)/\sigma_0(V)| \leq 1$ which means that the total conductivity $\sigma_0(V) + \sigma_1(V) \cos \varphi \geq 0$.

FIG. 2. — The ratio $\sigma_1(V)/\sigma_0(V)$ for the BCS value of the gap parameter $\Delta(T)$. $\sigma_1(V)/\sigma_0(V) \rightarrow 1$ as $V \rightarrow 0$.

When the junction is biased at zero voltage φ is constant and as seen from the figures 1 and 2 the σ_0 and σ_1 terms vanish. Thus the tunneling current is $I = j_1(0) \sin \varphi_0$ indicating that a finite dc current can flow through the junction without any voltage across it.

FIG. 3. — The increment of the ac-current amplitude when the $\cos \varphi$ term is included in percent of $j_1(V)$. Parameter : $V/V_g(T=0)$.

When the voltage across the junction is finite and pure dc, the $\sin \varphi$ and $\cos \varphi$ terms average to zero since φ varies linearly in time. Therefore only $\sigma_0(V)$ V contributes to the dc current when the voltage is non-zero.

Experimentally dc currents are observed for all voltages and thus the $\sin \varphi$ and σ_0 term have been experimentally verified.

2. Influence of σ_1 on the ac-current amplitude. —

From eq. (1.2) it follows that in addition to the dc-current $\sigma_0(V) V$ there is an ac-current with amplitude $\sqrt{j_1^2(V) + \{\sigma_1(V) V\}^2}$ and oscillating with the so-called Josephson frequency $2 |e| V/\hbar$. We will now discuss the possibility of verifying the presence of the $\cos \varphi$ term by measuring the ac-current amplitude when the junction is dc-voltage biased. If the $\cos \varphi$ term were not present the amplitude should be just $j_1(V)$ and thus a measurement of the current amplitude would serve as a verification of $\sigma_1(V)$ only if the difference between $j_1(V)$ and $\sqrt{j_1^2(V) + \{\sigma_1(V) V\}^2}$ is detectable i. e. greater than, say 20 %.

Experimentally there are two ways to proceed. The first is to fix the temperature and vary the voltage but this technique demands a wide-band detecting apparatus since the Josephson frequency varies 483.6 MHz/ μ V. The other procedure, to fix the voltage and vary the temperature, is preferable since the detection takes place only at one frequency and thus the use of tuned circuits, for example, becomes possible. For this reason the difference between the two possible amplitudes — shown in figure 3 — has been calculated as a function of temperature and with the voltage as parameter. As seen the difference is greater than 20 % for voltages above the zero temperature gap voltage. Unfortunately the increment is here almost temperature-independent and thus an absolute measurement is necessary to distinguish the two cases since they will have the same appearance, as shown in figure 4. If only a relative measurement of the current amplitude is possible, the percentage difference must therefore have an appreciable dependence on temperature. This happens for voltages below the zero temperature gap voltage but in order to have a maximum variation of 20 % only voltages in $0.8 \lesssim V/V_g(T=0) \lesssim 1$ can be used. Unfortunately the variation takes place in the high temperature region where the absolute value of the amplitude vanishes. Thus only voltages just below the zero temperature gap voltage remain. The current amplitudes corresponding to $V = V_g(T=0)$ and

$$V = 0.9 V_g(T=0)$$

are also shown in figure 4 and it is seen that although including the $\cos \varphi$ term gives a percentage variation from 0 to 20 % the shapes of the two curves are so much alike that it is doubtful whether a relative measurement is able to distinguish the two cases.

FIG. 4. — The ac-current amplitude I with and without the $\cos \varphi$ term included. Parameter: V/V_g ($T = 0$).

From this analysis it must be concluded that a relative measurement of the ac-current amplitude in a dc-voltage biased junction cannot be used, in practice, to verify the presence of the $\cos \varphi$ term. It serves however, as a new way of observing the temperature and voltage dependence of the Riedel-peak [8] i. e. the singularity of $j_1(V)$ at the gap voltage.

3. Experimental observation of the $\cos \varphi$ -term. — The expression for the tunneling current eq. (1.2) holds only for dc-voltages. In the case of a time-dependent voltage it becomes much more complicated see e. g. reference [6].

In the adiabatic approximation i. e. by assuming that the junction is able to respond instantaneously to the applied voltage we can however use eq. (1.2) if V is interpreted as the actual voltage and we take $d\varphi/dt = 2|e|\hbar V(t)/\hbar$.

If in addition we approximate $\sigma_0(V)$, $\sigma_1(V)$, and $j_1(V)$ by constants and introduce a current component due to the capacitance between the two superconductors forming the junction then the current is

$$I(t) = C \frac{dV}{dt} + j_1 \sin \varphi + \{ \sigma_0 + \sigma_1 \cos \varphi \} V. \quad (3.1)$$

Let us consider a simple example. At zero temperature the conductivities σ_0 and σ_1 vanish for voltages below the gap voltage. Suppose φ is almost constant i. e. $\varphi(t) = \varphi_0 + \delta\varphi(t)$, $|\delta\varphi| \ll 1$ then the dc-voltage will be zero since $V(t) = \delta\dot{\varphi}\hbar/2|e|$. Inserting this in eq. (3.1) and expanding $\sin \varphi$ and $\cos \varphi$ to first order in $\delta\varphi$ one obtains when the junction is dc-current biased

$$I_{dc} = \frac{\hbar C}{2|e|} \delta\ddot{\varphi} + j_1 \cos \varphi_0 \delta\dot{\varphi} + j_1 \sin \varphi_0. \quad (3.2)$$

A solution to this differential equation is

$$I_{dc} = j_1 \sin \varphi_0 \quad (3.3)$$

$$\delta\varphi = \varphi_1 \cos \omega_p t \quad (3.4)$$

$$\omega_p^2 = \frac{2|e|\hbar j_1}{\hbar C} \cos \varphi_0. \quad (3.5)$$

This means that once set up a small ac-voltage will without any applied signal remain in the junction if it has a frequency $\omega_p/2\pi$, the plasma frequency.

At finite temperatures the σ_0 and σ_1 terms in eq. (3.1) introduce a damping and therefore to maintain an oscillation an external ac-current source is needed but changing its frequency f the induced voltage will still exhibit a resonance when $f = f_p$.

A $\cos \varphi$ dependent current component was first observed in an experiment on this plasma resonance. The experiment was performed by N. F. Pedersen *et al.* [4], [5] who supplied a junction with an rf-current and measured the induced rf-voltage. Due to the non-linearity of the system also voltages at harmonics of the applied frequency f are induced and for convenience the *second* harmonic was measured.

The theoretical analysis of the plasma resonance experiment follows the lines of the example and yields

$$\frac{f_p}{Q} = \frac{\sigma_0}{2\pi C} \left[1 + \frac{\sigma_1}{\sigma_0} \left(\frac{f_p}{f_j} \right)^2 \right] \quad (3.6)$$

where $f_j = f_p$ ($I_{dc} = 0$) and Q is the quality factor of the resonance. From this expression it follows that plotting f_p/Q versus f_p^2 one should obtain a straight line which if the $\cos \varphi$ term (i. e. σ_1) were not present would have zero slope.

The experimental result of N. F. Pedersen *et al.* [4], [5] shown in figure 5 clearly exhibits a finite slope.

FIG. 5. — f_p/Q as measured by N. F. Pedersen *et al.* [4], [5]. Interpolation was performed disregarding the peak at $f_p \approx 10$ GHz which was ascribed to a geometrical resonance.

From this it was concluded that a $\cos \varphi$ term is present and they found $\sigma_1/\sigma_0 = -0.9 \pm 0.1$.

Theoretically this ratio is positive for small voltages as seen from figure 2. By insertion of the parameters corresponding to the experimental situation we find a value which — apart from the sign — is in agreement with the experimental value.

Thus there is disagreement between theory and experiment. Our theoretical expression for $\sigma_1(V)$ is in agreement with those of other authors [6]-[9].

Recently new evidence for a $\cos \varphi$ dependent term has been obtained by Falco *et al.* [10] who measured I - V characteristics of weak links. To interpret their experimental results they used the current-voltage relation eq. (3.1) but included on the right hand side also a current component due to thermal noise. For small values of the capacitance C one usually obtains a hyperbolic-like I - V curve but as indicated on the insert of figure 6 this is changed by the thermal noise. By measuring the slope at $I = 0$ Falco *et al.* [10] have obtained the result shown in figure 6 and com-

paring to the theoretical curves they conclude that $\sigma_1/\sigma_0 = -0.85 \pm 0.15$.

4. Model calculations of I - V curves. — The influence of the $\cos \varphi$ term on I - V characteristics has in the absence of thermal noise been examined by model calculations. Assuming $\sigma_0 \sim V$ and $\sigma_1/\sigma_0 = \text{constant}$ N. F. Pedersen and K. Saermark [12] have shown the problem to be analytically solvable but in the range of parameters considered they found that the I - V curve was only slightly affected by the presence of the $\cos \varphi$ term.

Setting $I(t) = I_{dc}$ W. Schlup [13] has by numerical solution of eq. (3.1) found that in some ranges of parameters the effect of the $\cos \varphi$ term may be observable. Using the result of W. Schlup one should however be aware of the approximations made. As seen from figure 2 $\sigma_1/\sigma_0 = \text{constant}$ is a crude approximation and it is usable at best only when the maximum voltage — during one period of oscillation — is below the gap voltage. This constraint can cause the calculation to be valid only in ranges of parameters where the effect is undetectable.

5. Summary. — We have presented curves showing the temperature and voltage dependence of $\sigma_1(V)$ and $\sigma_1(V)/\sigma_0(V)$ for a tunnel junction made of identical superconductors. By a theoretical analysis it has been shown that the $\cos \varphi$ term cannot be verified in practice from a relative measurement of the ac-current amplitude when the junction is dc-voltage biased.

Two recent experiments where a $\cos \varphi$ dependent term appears have been considered and it is found that the experimental values of σ_1/σ_0 have sign opposite to that predicted theoretically. In addition we have briefly discussed two I - V characteristic model calculations involving a $\cos \varphi$ dependent term.

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FIG. 6. — Experimental and theoretical results for dV/dI ($I \rightarrow 0$) in the presence of thermal noise (after Falco *et al.* [10]). Inserted: the I - V characteristic for different strengths of thermal noise (schematic after Ambegaokov and Halperin [11]).

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