THEORY OF FREQUENCY MIXING (*)

C. S. CHANG and P. STEHLE

Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA

Résumé. — Nous développons une méthode de la théorie quantique des champs pour décrire le mélange de fréquences au moyen de jonctions Josephson ou au moyen d'un faisceau d'électrons traversant une succession de cavités hyperfréquences. Un de nos buts est de discuter le mécanisme de base de la modulation du courant électronique et du rayonnement qui en découle dans un contexte de mécanique quantique. Nous utilisons le modèle de Heisenberg dans lequel les opérateurs de champ dépendent du temps et les vecteurs d'états sont fixes. Il est essentiel de ne tenir compte des valeurs données par la mécanique quantique que pour les quantités qui sont réellement observées. Ainsi, dans le cas du mélange de fréquences, on n'évalue pas le courant mais plutôt la puissance rayonnée qui est mesurable. Une évaluation du courant détruirait les corrélations entre les composantes de Fourier du courant produit dans la première cavité, ou celles du courant de la première jonction. En considérant le courant comme un opérateur, nous pouvons relier les corrélations réalisées dans la première cavité (jonction) au rayonnement dans la seconde. Un autre but de l'étude est de trouver une dépendance possible entre le rayonnement de sortie et l'état quantique du rayonnement incident préexistant dans les cavités ou les jonctions Josephson. Dans le cas d'un mélange de fréquences utilisant 2 cavités hyperfréquences où les champs sont considérés comme indépendants, nous montrons qu'il est possible de distinguer le cas où les 2 cavités sont dans un état cohérent de phase définie de celui où les états sont des états propres de l'énergie, par la valeur de la constante de renormalisation de la fonction d'onde. Cependant, si la phase relative des 2 cavités ne peut être définie, cette distinction disparaît. Ce résultat indique que le rayonnement est émis à la fréquence de battement même lorsque les champs des cavités sont pris comme états propres d'énergie. Dans ces conditions, aucune phase que ce soit ne peut être associée aux champs des cavités et le modèle classique de modulation de vitesse ne peut être appliqué.

Abstract. — We develop a quantum field-theoretic method to describe the effect of frequency mixing using Josephson junctions or using an electron beam traversing a succession of microwave cavities. One of our purposes is to discuss the basic mechanism of current modulation and the subsequent radiation in the complete quantum mechanical context. We use the Heisenberg picture, so that the field operators depend on time and state vectors are fixed. It is essential to take quantum mechanical expectation values only for the quantities that are actually observed. Thus, in the case of frequency mixing, we do not evaluate the current but rather the radiated power, which is observable. An evaluation of the current would destroy correlations between Fourier components of the current produced in the first cavity or in the first junction. Leaving the current as an operator permits the correlations produced in the first cavity (junction) to affect the radiation in the second cavity (junction). Another purpose of the approach is to find the possible dependence of the output radiation on the quantum state of the incident radiation preexisting in cavities or in Josephson junctions. In the case of frequency mixing using two microwave cavities where the fields are treated as independent, we show that it is possible to distinguish the case where both cavities are in definite phase-certain coherent states from the phase-uncertain number states by the value of the wave function renormalization constant. However, if the relative phase of the two cavities cannot be defined, this distinction disappears. This indicates that the radiation occurs at the beat frequency even when the cavity fields are taken to be in energy eigenstates, so that no phase whatever can be associated with the fields in those cavities and thus a classical velocity modulation picture does not apply.

1. Introduction. — Frequency mixing occurs in so many circumstances is and used in so many practical devices that a discussion of the theory of frequency mixing may seem superfluous. When the frequency mixing involves the presence of two currents, one characterized by one frequency and the second by another frequency, any non-linear process such as radiation will lead to the generation of sum

and difference frequencies. We are concerned here with a quantum field-theoretic description of frequency mixing. A minor reason for being so concerned is simply to see how to do it ; any process whatever possessing a physical description should have a quantum field-theoretic one. Our major reason is, however, more practical. We want to find out to what extent the phenomenon of frequency mixing depends on the quantum state of the system, as distinct from depending on the dynamics. In particular, must one be able to define a phase for each

(*) Work supported in part by the US Army Research Office.

of the signals to be mixed in order for detectable mixing to occur ? If two oscillators, each initially in an energy eigenstate and therefore not giving rise to any expectation value of the current, are coupled to the electromagnetic field, will they radiate at mixed frequencies ? Or does frequency mixing occur only if the oscillators are in coherent states or something similar ?

In Josephson junctions the currents involved are not describable accurately by classical dynamics. It is in understanding the radiative properties of such junctions that quantum theory may be first needed. The paper which follows is, nonetheless, phrased in terms of electron beams and cavities such as are present in klystrons. This does not affect the principal features of the discussion, but it does give a simple example of the kind of system of interest which can be worked out rather completely. The analogy between Josephson junctions and klystrons is fairly close. An essential parameter in a klystron [1] is the bunching parameter which depends on a phase difference between the currents in two cavities. An essential parameter in Josephson junctions is the phase difference [2] between the long-range order parameters in the two superconductors. It appears that these two quantities play similar roles, and permit the application of results obtained for klystrons to Josephson junctions.

The results of the analysis of frequency mixing given here are that frequency mixing occurs even when the phases of the two currents being mixed are completely undefined. There is a term in the expression for the radiated power which depends on the initial state of the mixed fields, and which does not vanish and if both fields are in coherent states with well-defined phases. Detection of the contribution of this term seems unlikely under experimental conditions.

2. Field-theoretic formulation. — The fundamental equations of quantum electrodynamics [3] apply to all electromagnetic systems. We write them down in the Heisenberg picture and then give explicit expressions for the interaction of concern here, an electron interacting with the field in a resonant cavity. The Heisenberg picture is convenient because it makes the operator equations of motion for the electron and electromagnetic field have the same form as the classical equations with the interaction included, and permits the evaluation of physical quantities such as power radiated at a given frequency in terms of the initial state.

The electron field operator is expanded in powers of the electron charge e . Assuming only the z coordinate to be needed,

$$\psi^{\text{out}}(z, t) = \sum_{n=0}^{\infty} e^n \psi^{(n)}(z, t). \quad (1)$$

The n th order operator $\psi^{(n)}(z, t)$ is then given by

$$\psi^{(n)}(z, t) = i^n \int_{-\infty}^t dt_1 dz_1 \dots \int_{-\infty}^{t_{n-1}} dt_n dz_n \times \\ \times [V(x_n), [\dots, [V(x_1), \psi^{\text{in}}(x)] \dots]]. \quad (2)$$

Here

$$V(x) = V(z, t) = -j_{\mu}^{\text{in}}(x) A_{\mu}^{\text{in}}(x), \quad (3)$$

$$j_{\mu}^{\text{in}}(x) = \frac{i}{2} [\bar{\psi}^{\text{in}}(x), \gamma_{\mu} \psi^{\text{in}}(x)]. \quad (4)$$

The $V(x)$ in (3) contains in principle all interactions, including self-interactions. We need not explicitly allow for self-interactions as at the frequencies involved here they can be included simply by using the observed (renormalized) values of the electron charge and mass. Only the interaction with the cavity field is explicitly taken into account.

The electron is initially a free electron,

$$\psi^{\text{in}}(x) = \frac{1}{\sqrt{L}} e^{-iE_0 t + i p_0 z} b_{\text{in}}, \quad (5)$$

where b_{in} is an electron annihilation operator. The cavity field is assumed to consist of a single mode

$$A_{\mu}^{\text{in}}(x) = (2\omega)^{-1/2} e_{\mu} [U_{\omega}(z) e^{-i\omega t} a_{\text{in}} + hc]. \quad (6)$$

Eq. (5) and (6) specify our problem.

In the Heisenberg picture the n th order term (2) is most easily written in terms of retarded products,

$$\psi^{(n)}(z, t) = \\ = \frac{1}{n!} \int_{-\infty}^t dt_1 \int_{-d_0/2}^{d_0/2} dz_1 \dots \int_{-\infty}^t dt_n \int_{-d_0/2}^{d_0/2} dz_n \times \\ \times R(x, x_1, x_2, \dots, x_n) \quad (7)$$

where the upper limits of the t -integrations are now all t , and the spatial integration is only over the extent d_0 of the cavity.

There is a reduction formula [4] for R -products similar to that of time ordered or T -products. The analysis of (7) is formally identical to that of T -products in terms of the Feynman diagram. The field $A_{\mu}^{\text{in}}(x)$ is a cavity field, not a field in free space, so that it is not invariant under translation and momentum need not be conserved in the interaction $V(x)$. It is possible, therefore, for both the incoming and outgoing electron lines to be free-electron lines, even when a finite amount of energy is emitted or absorbed at a vertex. It is assumed that the electron remains on the mass shell after each interaction. Applying

the R -product reduction formula to (7) and neglecting all terms containing electron loops and internal photon lines (electron-electron interaction), we obtain

$$R(x; x_1, x_2, \dots, x_n) = (-i)^n K_{\mu_1 \mu_2 \dots \mu_n}(x; x_1, x_2, \dots, x_n) \psi^{\text{in}}(x) : A_{\mu_1}^{\text{in}}(x_1) A_{\mu_2}^{\text{in}}(x_2) \dots A_{\mu_n}^{\text{in}}(x_n) : \quad (8)$$

where colons denote the normal product of the enclosed factors, and

$$K_{\mu_1 \mu_2 \dots \mu_n}(x; x_1, x_2, \dots, x_n) = S_R(x - x_1) \gamma_{\mu_1} S_R(x_1 - x_2) \gamma_{\mu_2} \times \dots \times S_R(x_{n-1} - x_n) \gamma_{\mu_n} \quad (9)$$

with $S_R(x - y)$ denoting the free-electron retarded propagator.

The assumption that the electron remains on the mass shell means that only pole contributions are retained in the integrals of (7). Using this approximation, (7) becomes

$$e^n \psi^{(n)}(z, t) = \frac{(-i)^n}{n! \sqrt{L}} \sum_m \binom{n}{m} \lambda^n \exp \{ -i[E_0 t + (n - 2m) \omega t] \} \times \exp[i(P_0 + \Delta P_{n-2m}) z] (a_{\text{in}}^\dagger)^m (a_{\text{in}})^{n-m} b_{\text{in}} \quad (10)$$

for z beyond the input cavity, i. e., $z > d_0/2$, and with

$$\Delta P_n = P_0 \left[\left(1 + n \frac{\omega}{E_0} \right)^{1/2} - 1 \right] \approx P_0 \left[n \frac{\omega}{2 E_0} - \frac{n^2}{8} \left(\frac{\omega}{E_0} \right)^2 \right],$$

$$\lambda = \frac{e\beta \tilde{V}_1}{\omega}, \quad \beta = \sin \frac{(\omega d_0/2 v_0)}{(\omega d_0/2 v_0)}. \quad (11)$$

Here $\tilde{V}_1$ is the potential across the gap arising from considering the mode function $U_\omega(z)$ as a classical field. $\tilde{V}_1$ depends on the geometry of the gap through the normalization of $U_\omega(z)$; it does not depend on the level of excitation of the cavity, this entering through either the photon number of the initial cavity state or other similar parameter. β is the beam coupling coefficient [1] which is unity if the transit time of the electron across the gap is negligible and is otherwise less than unity. The outgoing electron field operator after the input cavity can now be written,

$$\psi^{\text{out}}(z, t) = \sum_{n_1, n_2} \frac{(-i\lambda)^{n_1+n_2}}{n_1! n_2!} \exp \{ i[(n_1 - n_2) \omega t + \Delta P_{n_1-n_2} z] \} \times (a_{\text{in}}^\dagger)^{n_1} (a_{\text{in}})^{n_2} b_{\text{in}} \frac{\exp(-iE_0 t + iP_0 z)}{\sqrt{L}}. \quad (12)$$

This is an operator equation.

In the case of two excited cavities separated by a distance s_1 , (6) now becomes,

$$A_\mu^{\text{in}}(x) = (2\omega_1)^{-1/2} \varepsilon_\mu^{(1)} U_{\omega_1}(z) e^{-i\omega_1 t} a_1 + (2\omega_2)^{-1/2} \varepsilon_\mu^{(2)} U_{\omega_2}(z + s_1) e^{-i\omega_2 t} a_2 + hc \quad (13)$$

then, instead of (12), we have

$$\psi^{\text{out}}(z, t) = \frac{1}{\sqrt{L}} \sum_{\{n\}} \frac{(-i\lambda_1)^{n_1}}{n_1!} \frac{(-i\lambda_1)^{n'_1}}{n'_1!} \frac{[-i\lambda_2^{(-)}(n_1 - n'_1)]^{n_2}}{n_2!} \frac{[-i\lambda_2^{(+)}(n_1 - n'_1)]^{n'_2}}{n'_2!} \times$$

$$\times \exp \{ -i[E_0 + (n_1 - n'_1) \omega_1 + (n_2 - n'_2) \omega_2] t \} \exp[i(P_0 + \Delta P_{n_1-n'_1, n_2-n'_2}) z]$$

$$\times (a_1^\dagger)^{n_1} (a_1)^{n'_1} (a_2^\dagger)^{n_2} (a_2)^{n'_2} b_{\text{in}}, \quad (14)$$

where $\{n\} = (n_1, n'_1, n_2, n'_2)$, and

$$\lambda_2^{(\pm)}(n) = \lambda_2 \exp(is_1 \Delta P_{n, \pm 1}), \quad (15)$$

$$\Delta P_{n_1, n_2} = P_0 \left[\left(1 + n_1 \frac{\omega_1}{E_0} + n_2 \frac{\omega_2}{E_0} \right)^{1/2} - 1 \right], \quad (16)$$

$$\Delta P_{n, \pm 1} = P_0 \left[\left(1 + n \frac{\omega_1}{E_0} \right)^{1/2} - \left(1 + n \frac{\omega_1}{E_0} \pm \frac{\omega_2}{E_0} \right)^{1/2} \right], \quad (17)$$

$$\lambda_i = \frac{e\beta_i \tilde{V}_i}{\omega_i}, \quad i = 1, 2. \quad (18)$$

Here λ_i and β_i are the obvious generalization of λ and β for one cavity defined in (11). The application of (14) to the frequency mixing will be given in Section 4.

3. Renormalization of the outgoing state. — When two cavities affect the electron as described in the previous section, the electron entering the second cavity is no longer described by the simple free particle operator (5) but by one including the effect of the first cavity. It is then necessary to introduce a renormalization constant Z_2 in order to preserve the total charge, or to conserve electrons. This occurs because a variety of intermediate electron energies can contribute to any given final electron energy, unlike the single-gap situation, and the outgoing operator is not automatically normalized in an unrenormalized single-electron state. We therefore define the « out » state by [5]

$$| \text{out} \rangle = Z_2^{1/2} | 1 \rangle_{\text{in}} | R \rangle_{\text{in}} \quad (19)$$

where $| 1 \rangle_{\text{in}}$ is the incoming electron and $| R \rangle_{\text{in}}$ the incoming radiation state, and require that

$$\langle \text{out} | \int_{-\infty}^{\infty} dz \psi^{\text{out}\dagger}(z, t) \psi^{\text{out}}(z, t) | \text{out} \rangle = 1. \quad (20)$$

It can be verified that when only one cavity is present we have

$$Z_2 = 1 \quad (21)$$

for any kind of cavity state $| R \rangle_{\text{in}}$. For example, $| R \rangle_{\text{in}}$ may be either the number state $| N \rangle$ or the coherent state $| \alpha \rangle$.

When two cavities separated by the distance s_1 are present, however, we have

$$Z_2^{-1} = I_0(| \xi_2 | \sin \eta_1), \quad (22)$$

if $| R \rangle_{\text{in}}$ is one of the three states, $| N_1, N_2 \rangle$, $| N_1, \alpha_2 \rangle$ and $| \alpha_1, N_2 \rangle$, as shown in reference [6]. Here $I_0(x)$ is the modified Bessel function of order zero, and

$$\xi_2 = \frac{2 \beta_2 e V_2}{\omega_2} e^{i\eta}, \quad \eta_1 = \frac{1}{8} P_0 s_1 \left(\frac{\omega_1}{E_0} \right)^2. \quad (23)$$

V_1 and V_2 are the peak potential drop across cavity 1 and 2 respectively. ($V_1 = N_1 \tilde{V}_1$ if cavity field state is in number state, or $V_1 = | \alpha_1 |^2 \tilde{V}_1$, if it is in the coherent state). On the other hand, if

$$| R \rangle_{\text{in}} = | \alpha_1, \alpha_2 \rangle, \quad (24)$$

then Z_2 depends on the relative phase α_1 and α_2 . If the two cavities are excited by two uncorrelated sources, then such a dependence will disappear after averaging over the phases of α_1 and α_2 , and (22) is again obtained.

Denoting the form of Z_2 with the use of (24) by $\tilde{Z}_2$, we have, after some manipulations involving

Graf's form of the addition theorem, as demonstrated in Chang and Stehle [6],

$$\begin{aligned} Z_2^{-1} &= I_0(| \xi_2 | \sin \eta_1) + \\ &+ 2 \sum_{m=1}^{\infty} J_m(2 \xi_1 \sin m \eta_1) J_{km}(\tilde{w}_m) \times \\ &\times \text{Re} \left[e^{im(\theta_1 - k\theta_2) - 2im\varphi_1 - im^2\eta_1 + im\pi/2} \right. \\ &\times \left. \left(\frac{e^{-2i\eta_1} - e^{im\eta_1}}{e^{-2i\eta_1} - e^{-im\eta_1}} \right)^m \right], \end{aligned} \quad (25)$$

where

$$\begin{aligned} k &= \frac{\omega_1}{\omega_2}, \quad \varphi_1 = \frac{\omega_1 s_1}{v_0}, \quad \alpha_i = | \alpha_i | e^{i\theta_i}, \quad i = 1, 2, \\ \xi_1 &= \frac{2 \beta_1 e V_1}{\omega_1}, \quad \tilde{w}_m = \sqrt{2} | \xi_2 | (\cos \eta_1 - \cos m \eta_1)^{1/2}. \end{aligned} \quad (26)$$

This is the only case where a relative phase of the two input signals can be defined, and the renormalization here does depend on the phase difference $\theta_1 - k\theta_2$. In practice it is not likely that this difference will be strictly time independent, and if it is not, the second term in (25) will have to be averaged and the previous result is again obtained.

4. Calculation of power output at mixed frequency. —

We consider the frequency mixing experiments to consist of three cavities in succession. The first two cavities are used for modulation and the third cavity is a detector to be tuned to a fixed frequency ν .

To compute the frequency-mixing effect, we calculate the current density operator from (14). It follows from its definition (4), and can be written as

$$\hat{j}^{\text{out}}(z, t) = \sum_{n_1, n_2} \hat{j}_{n_1, n_2}(z) e^{-in_1\omega_1 t - in_2\omega_2 t}. \quad (27)$$

The current (27) is a source for the field operator $A^{\text{out}}(z, t)$, being related to it through a retarded photon propagator $D_R(z, z'; t, t')$ coming from Maxwell's equations [7]. If we restrict our attention to radiation in the third cavity resonant at frequency ν , the retarded photon propagator can be written in terms of the mode function $U_\nu(z)$:

$$D_R(z, z'; t, t') = \frac{1}{2 i \nu} U_\nu(z) U_\nu^*(z') e^{-i\nu(t-t')} \theta(t-t'), \quad (28)$$

where $\theta(t)$ is the unit step function; we have

$$\begin{aligned} \hat{A}^{\text{out}}(z, t) &= \\ &= \int_{s_2 - (d_2/2)}^{s_2 + (d_2/2)} dz' \int_{-\infty}^{\infty} dt' D_R(z, z'; t, t') \hat{j}^{\text{out}}(z', t'), \end{aligned} \quad (29)$$

where s_2 is the distance between first and third cavities.

The power output at the third cavity is

$$P_{\text{out}} = v^2 \left\langle \text{out} \left| \int_{s_2 - (d_2/2)}^{s_2 + (d_2/2)} dz : \hat{A}^{\text{out}}(z, t) \hat{A}^{\text{out}}(z, t) : \right| \text{out} \right\rangle. \quad (30)$$

Looking at the time integration of (29), we easily see that a resonant denominator of the form

$$v - (n_1 \omega_1 + n_2 \omega_2)$$

appears in (30). The power delivered by the electron to the third cavity tuned at $v \approx n_1 \omega_1 + n_2 \omega_2$ is therefore determined by the quantity

$$P_{n_1 n_2} = \langle \text{out} | \hat{j}_{n_1, n_2}^{(+)}(s_2) \hat{j}_{n_1, n_2}^{(-)}(s_2) | \text{out} \rangle, \quad (31)$$

where

$$\hat{j}_{n_1, n_2}^{(+)}(s_2) = \frac{1}{(2v)^{1/2}} \int_{s_2 - (d_2/2)}^{s_2 + (d_2/2)} dz \hat{j}_{n_1, n_2}(z) U_v(z), \quad (32)$$

$$\hat{j}_{n_1, n_2}^{(-)}(s_2) = [\hat{j}_{n_1, n_2}^{(+)}(s_2)]^\dagger.$$

Using (19), (31) can be written as

$$P_{n_1 n_2} = Z_2 \langle R | \hat{j}_{n_1, n_2}^{(+)}(s_2) \hat{j}_{n_1, n_2}^{(-)}(s_2) | R \rangle_{\text{in}}, \quad (33)$$

where Z_2 is the wave function renormalization constant shown in Section 3.

It can be shown that the expectation value indicated in (33) does not depend on the quantum state $|R\rangle_{\text{in}}$ of the first two cavities. From (14), (27) and (32), and after some algebra, we get

$$P_{n_1 n_2} = Z_2 |J_{n_1}(w_{n_1 n_2}^1) J_{n_2}(w_{n_1 n_2}^2)|^2, \quad (34)$$

where

$$\begin{aligned} w_{n_1 n_2}^1 &= 2 \xi_1 \sin \left[\frac{1}{2} (2 n_1 \eta'_1 + 2 n_2 \eta'_0 - n_2 \eta_1) \right], \\ w_{n_1 n_2}^2 &= \sqrt{2} |\xi_2| [\cos 2 \eta_1 - \cos (n_1 \eta_1 - 2 n_1 \eta'_0 - 2 n_2 \eta'_2)]^{1/2}, \\ \eta'_1 &= \frac{1}{8} P_0 s_2 \left(\frac{\omega_1}{E_0} \right)^2, \quad \eta'_0 = \frac{1}{8} P_0 s_2 \frac{\omega_1 \omega_2}{E_0^2}. \end{aligned} \quad (35)$$

If the first two cavities are separated by a small distance such that

$$s_1 = 0 \quad \text{or} \quad \eta_1 = 0, \quad (36)$$

we have

$$\begin{aligned} w_{n_1 n_2}^1 &= 2 \xi_1 \sin (n_1 \eta'_1 + n_2 \eta'_0), \quad w_{n_1 n_2}^2 = 2 \xi_2 \sin (n_2 \eta'_2 + n_1 \eta'_0), \\ Z_2 &= 1. \end{aligned} \quad (37)$$

If $\eta'_i, \eta'_0 \ll 1$, (37) can be expanded and finally written in terms of bunching parameters [1] and the classical results are obtained for the power output at $v \approx n_1 \omega_1 + n_2 \omega_2$. Since experiments on beating the high-order harmonics of klystron radiation with far-infrared maser radiation [8] are now performed using Josephson junctions, it is possible to achieve conditions where $n_i \eta'_i$ and $n_i \eta'_0$ are no longer much less than unity and where appreciable quantum effects can be expected. In this connection, we may note that the inter-cavity separation s_1 seems to have an analog in the case of Josephson junctions where the phase difference of superconducting electrons in

the superconductors will make the condition $\eta_1 = 0$ impossible. It is thus desirable to apply the complete expression (34) and to identify the parameters appearing there with those describing a Josephson junction.

5. Discussion. — The theory given here is a straightforward application of quantum physics to a problem that is usually treated by classical physics. The results are of some interest as they show the existence of circumstances under which the classical theory does not apply, while for the commonly met circumstances the results are indistinguishable from the classical ones. When units are used in which $\hbar$ is not unity,

the quantity ξ_1 defined in (26) contains a factor $1/\hbar$ and the quantity η'_i defined in (35) contains a factor $\hbar$. Planck's constant cancels out of the quantity w of (35) when η'_i is small and the harmonic numbers are not too large, yielding the classical result [1]. For not so small η'_i and sufficiently high harmonics it is not

permissible to use small angle expansions and Planck's constant remains explicitly present.

The radiation emitted has been evaluated in terms of parameters appropriate to electron beams and cavities. An evaluation in terms of parameters appropriate to Josephson junctions will be given later.

References

- [1] HARRISON, A. E., *Klystron Tubes* (McGraw-Hill, New York, 1947).
- [2] JOSEPHSON, B., *Adv. Phys.* **14** (1965) 419.
- [3] HAGEDORN, R., *Introduction to Field Theory and Dispersion Relations* (Pergamon, New York) 1964.
- [4] *Ibid.*, p. 27-29.
- [5] ROMAN, P., *Introduction of Quantum Field Theory* (John Wiley, New York) 1969.
- [6] CHANG, C. S. and STEHLE, P., *Phys. Rev. A* **8** (1973) 318.
- [7] JACKSON, J. D., *Classical Electrodynamics* (John Wiley, New York) 1965.
- [8] MACDONALD, D. G., RISLEY, A. S., CUPP, J. D. and EVENSON, K. M., *Appl. Phys. Lett.* **18** (1971) 162.
- [9] MCDONALD, D. G., RISLEY, A. S., CUPP, G. D., EVENSEN, K. M. and ASHLEY, G. R., *Appl. Phys. Lett.* **20** (1972) 296.