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Gaussian propagators for the Ising spin glass below T_c

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Résumé. — Comme élément d'une théorie des champs à construire autour de la solution de champ moyen de Parisi pour un verre de spin d'Ising, nous donnons l'ensemble complet des propagateurs au niveau gaussien.

Abstract. — As building blocks of a field theory to be constructed upon Parisi's mean field solution for an Ising spin glass, the complete set of the Gaussian propagators is given.

Having passed the stability test [1] and acquired a clear physical interpretation [2] Parisi's mean field theory [3] is now well established as a proper solution of the SK problem [4]. Despite considerable numerical effort the nature (or indeed the very existence) of a spin glass transition in the short range model [5] remains controversial [6], however. Guidance from analytic theory would evidently be of valuable help in this situation. As a first step beyond mean field theory, partial information on Gaussian fluctuations has become available recently [7-9], but a complete set of Gaussian propagators that would allow one to construct the field theory of the Ising spin glass has never been published. Our purpose here is to fill this gap. The Green function of a spin glass in the ordered phase is a complicated object and the formulae given below inevitably reflect this fact. We felt necessary to include detailed equations in order to make the results reproducible.

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As a simple illustration of their use, the « single-valley » contribution to the correlation function $(\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle)^2$ will be worked out at the end of this note.

Our starting point is the same truncated free energy functional [3], valid for $\tau = (T_c - T)/T_c \ll 1$ and external field h small, that was used for the determination of the fluctuation spectrum [1]. The free Green functions $G_{\alpha\beta,\gamma\delta}$ of the theory are obtained by inverting the stability matrix $M_{\alpha\beta,\gamma\delta}$ associated with that free energy functional. The equation $\mathbb{G} \cdot (p^2 + \mathbb{M}) = \mathbb{I}$ yields

$$G_{\alpha\beta,\gamma\delta}(p^2 - 2\tau - 2q_{\gamma\delta}^2) - \sum_{\omega \neq \gamma,\delta} (G_{\alpha\beta,\gamma\omega} q_{\omega\delta} + G_{\alpha\beta,\delta\omega} q_{\omega\gamma}) = \delta_{\alpha\gamma} \delta_{\beta\delta} \quad (1)$$

where p is the wave vector, $q_{\alpha\beta}$ is the stationary point of the free energy functional, and $\alpha, \beta, \gamma, \delta$ are replica indices. Note that the magnetic field enters only through $q_{\alpha\beta}$.

The ultrametric structure [10] of Parisi's solution [3] for $q_{\alpha\beta}$ dictates the following parametrization for $G_{\alpha\beta,\gamma\delta}$:

$$G_{\alpha\beta,\gamma\delta} \rightarrow G_{z_1, z_2}^{x, y} \quad (2)$$

where $x = \alpha \cap \beta, y = \gamma \cap \delta, z_1 = \max(\alpha \cap \gamma, \alpha \cap \delta), z_2 = \max(\beta \cap \gamma, \beta \cap \delta)$ and the overlap $\alpha \cap \beta$ is equal to x if $q_{\alpha\beta} = q(x)$. G is symmetric in z_1, z_2 . Because of ultrametricity again, of the four numbers x, y, z_1, z_2 at most *three* can be different, and the various components of G can be given in terms of seven different functions. These satisfy a set of coupled integral equations that can be obtained by writing out (1) under the parametrization (2):

$$(p^2 - 2\tau - 2q^2(x)) G_{1,1}^{x,x} + 2 \int_0^x dt q(t) G_{1,x}^{x,t} + 2 x q(x) G_{1,x}^{x,x} + 2 \int_x^1 dt q(t) G_{1,t}^{x,x} + 2 q(x) \int_x^1 dt G_{1,x}^{x,t} = 1 \quad (3)$$

$$\frac{1}{2} p^2 G_{z,z}^{x,y} + \int_0^y dt q(t) G_{z,z}^{x,t} + q(y) \int_y^1 dt G_{z,z}^{x,t} + q(z) \left\{ z G_{z,z}^{x,z} + \int_z^x dt G_{t,t}^{x,z} + x G_{x,x}^{x,z} + 2 \int_x^1 dt G_{t,x}^{x,z} - 2 G_{1,x}^{x,z} \right\} = 0, \quad z < x, y \quad (4)$$

$$\frac{1}{2} p^2 G_{z,x}^{x,y} + \int_0^y dt q(t) G_{z,x}^{x,t} + q(y) \int_y^1 dt G_{z,x}^{x,t} + q(z) \left\{ z G_{z,x}^{x,z} + \int_z^1 dt G_{t,x}^{x,z} - G_{1,x}^{x,z} \right\} + q(x) \left\{ x G_{z,x}^{x,x} + \int_x^1 dt G_{z,t}^{x,x} - G_{z,1}^{x,x} \right\} = 0, \quad x < z < y \quad (5)$$

$$(p^2 + q^2(z) - q^2(y)) G_{z,z}^{x,y} + \int_0^y dt q(t) (G_{y,y}^{x,t} + G_{z,z}^{x,t}) + \int_y^z dt q(t) G_{t,t}^{x,y} + q(y) \left\{ \int_y^1 dt G_{y,y}^{x,t} + y G_{y,y}^{x,y} + \int_y^1 dt G_{z,z}^{x,t} + z G_{z,z}^{x,z} + \int_z^x dt G_{t,t}^{x,z} + x G_{x,x}^{x,z} + 2 \int_x^1 dt G_{t,x}^{x,z} - 2 G_{1,x}^{x,z} \right\} + q(z) \left\{ x G_{x,x}^{x,y} + \int_x^x dt G_{t,t}^{x,y} + 2 \int_x^1 dt G_{t,x}^{x,y} - 2 G_{1,x}^{x,y} \right\} = 0, \quad y < z < x \quad (6)$$

$$\begin{aligned}
& (p^2 + q^2(z) - q^2(y)) G_{z,x}^{x,y} + \int_0^y dt q(t) (G_{z,x}^{x,t} + G_{y,x}^{x,t}) + \int_y^z dt q(t) G_{t,x}^{x,y} + \\
& + q(y) \left\{ y G_{y,x}^{x,y} + \int_y^1 dt G_{y,x}^{x,t} + \int_z^1 dt (G_{z,x}^{x,t} + G_{t,x}^{x,z}) + z G_{z,x}^{x,z} + \int_y^z dt G_{z,x}^{x,t} - G_{1,x}^{x,z} \right\} \\
& + q(x) \left\{ x G_{y,x}^{x,x} + \int_x^1 dt G_{y,t}^{x,x} + x G_{z,x}^{x,x} + \int_x^1 dt G_{z,t}^{x,x} - G_{y,1}^{x,x} - G_{z,1}^{x,x} \right\} \\
& + q(z) \left\{ \int_z^1 dt G_{t,x}^{x,y} - G_{1,x}^{x,y} \right\} = 0, \quad x < y < z \quad (7)
\end{aligned}$$

$$\begin{aligned}
& (p^2 + q^2(z) - q^2(y)) G_{z,x}^{x,y} + \int_0^y dt q(t) (G_{y,y}^{x,t} + G_{z,x}^{x,t}) + \int_y^x dt q(t) G_{t,t}^{x,y} + \\
& + \int_x^z dt q(t) G_{t,x}^{x,y} + q(y) \left\{ \int_y^1 dt G_{y,y}^{x,t} + y G_{y,y}^{x,y} + \int_y^1 dt G_{z,x}^{x,t} + z G_{z,x}^{x,z} + \int_z^1 dt G_{t,x}^{x,z} \right. \\
& + x G_{z,x}^{x,x} + \left. \int_x^1 dt G_{z,t}^{x,x} - G_{1,x}^{x,z} - G_{z,1}^{x,x} \right\} + q(x) \left\{ x G_{x,x}^{x,y} + \int_x^1 dt G_{x,t}^{x,y} - G_{x,1}^{x,y} \right\} \\
& + q(z) \left\{ \int_z^1 dt G_{t,x}^{x,y} - G_{1,x}^{x,y} \right\} = 0, \quad y < x < z \quad (8)
\end{aligned}$$

$$(p^2 + q^2(z_1) + q^2(z_2) - 2q^2(x)) G_{z_1,z_2}^{x,x} + A(z_1, z_2) + A(z_2, z_1) = 0 \quad x < z_1, z_2 \quad (9)$$

with

$$\begin{aligned}
A(z_1, z_2) = & \int_0^x dt q(t) G_{z_1,x}^{x,t} + \int_x^{z_2} dt q(t) G_{z_1,t}^{x,x} + q(z_2) \left\{ \int_{z_2}^1 dt G_{z_1,t}^{x,x} - G_{z_1,1}^{x,x} \right\} + \\
& + q(x) \left\{ x G_{z_1,x}^{x,x} + \int_x^1 dt G_{z_1,t}^{x,x} + z_1 G_{z_1,x}^{x,z_1} + \int_{z_1}^1 dt G_{t,x}^{x,z_1} - G_{1,x}^{x,z_1} \right\}. \quad (10)
\end{aligned}$$

Equations (4) to (10) are closely related to the « third family equations » that determine the replicon band of the spectrum [1]. The solutions to equations (3) to (10) can naturally be split into a longitudinal-anomalous (LA) and replicon (R) contribution [8] :

$$G = G^{\text{LA}} + G^{\text{R}}.$$

The replicon contribution is relatively simple (see also [8, 9]) :

$$\begin{aligned}
(G_{1,1}^{x,x})^{\text{R}} = & \int_x^1 \frac{dk_1}{k_1} \int_x^1 \frac{dk_2}{k_2} \frac{\partial^2}{\partial k_1 \partial k_2} \frac{1}{p^2 + \lambda(x; k_1, k_2)} - \\
& - 2 \int_x^1 \frac{dk}{k} \frac{\partial}{\partial k} \frac{1}{p^2 + \lambda(x; 1, k)} + \frac{1}{p^2 + \lambda(x; 1, 1)} \quad (11)
\end{aligned}$$

$$(G_{1,z}^{x,x})^{\text{R}} = \int_x^1 \frac{dk_1}{k_1} \int_x^z \frac{dk_2}{k_2} \frac{\partial^2}{\partial k_1 \partial k_2} \frac{1}{p^2 + \lambda(x; k_1, k_2)} - \int_x^z \frac{dk}{k} \frac{\partial}{\partial k} \frac{1}{p^2 + \lambda(x; 1, k)}, \quad z > x \quad (12)$$

$$(G_{z_1, z_2}^{x, x})^R = \int_x^{z_1} \frac{dk_1}{k_1} \int_x^{z_2} \frac{dk_2}{k_2} \frac{\partial^2}{\partial k_1 \partial k_2} \frac{1}{p^2 + \lambda(x; k_1, k_2)}, \quad z_1, z_2 > x. \quad (13)$$

All other R components vanish. In equations (11) to (13) $\lambda(x; k_1, k_2) = q^2(k_1) + q^2(k_2) - 2q^2(x)$ is the « third family » eigenvalue [1].

The results obtained by Goltsev [11] for the R propagators at $x = 0$ (and in zero field) are in agreement with equations (11) to (13) and with the full propagators as derived by Sompolinsky and Zippelius [7] in the same limit (since for $x = 0$ the LA contribution vanishes [8] in zero field).

As for the LA contributions, their derivation is far more elaborate, and no closed form has been available so far. They are given by formulae of the following general structure :

$$(G_{z_1, z_2}^{x, y})^{LA} = - \int_0^{\max(z_1, z_2)} dk w(k) \frac{1}{k} \frac{\partial}{\partial k} F_k(u, v) + g F_1(u, v) \quad (14)$$

where $u = \min(x, y)$ and $v = \max(x, y)$.

If $\max(z_1, z_2)$ happens to be the smallest of the *three* different values of x, y, z_1, z_2 then the weight $w(k) \equiv 1$. If $\max(z_1, z_2)$ is the middle one of the three then the smallest falls on the integration interval, dividing it into two parts : $w(k) = 1$ over the lower part and $w(k) = 1/2$ over the upper one. If $\max(z_1, z_2)$ is the largest of the three, the other two divide the integration interval into three parts with $w(k) = 1, 1/2$, and $1/4$ over the lower, middle, and upper parts, respectively. The weight g in the surface term is $1/2$ for the « first propagator » ($z_1 = z_2 = 1$), $1/4$ for the « second propagator » (z_1 or z_2 equal to one, the other smaller), and zero for the « third propagator » (both z_1, z_2 less than one).

The function F_k was already given in [8]

$$F_k(u, v) = \frac{\phi_k^+(u) \phi_k^-(v)}{\Delta_k X_k(u) X_k(v)} \quad (15)$$

where

$$X_k(u) = \begin{cases} \frac{1}{2} (q^2(u) - q^2(k) - p^2), & u < k < x_1 \\ -\frac{1}{2} p^2, & k < u \end{cases} \quad (16)$$

and $\phi_k^\pm$ are the solutions of the integral equations

$$\phi_k^+(u) = q(u) + \int_0^u \frac{\Delta t}{X_k(t)} (q(t) - q(u)) \phi_k^+(t) \quad (17)$$

$$\phi_k^-(v) = 1 - \int_v^1 \frac{\Delta t}{X_k(t)} (q(t) - q(v)) \phi_k^-(t) \quad (18)$$

with the shorthand notation

$$\Delta t = \begin{cases} \frac{1}{2} dt, & t < k \\ \frac{1}{2} k \delta(t - k), & t = k \\ dt, & t > k \end{cases} \quad (19)$$

The quantity Δ_k in (15) is twice the Wronskian of $\phi_k^\pm$:

$$\Delta_k = 2(\dot{\phi}_k^+(u) \phi_k^-(u) - \phi_k^+(u) \dot{\phi}_k^-(u)) \quad (20)$$

independent of u .

Whenever k goes beyond the interval (x_0, x_1) of the lower and upper breakpoints of Parisi's $q(x)$, it is to be replaced by x_0 or x_1 .

By help of equations (10) to (20) one can construct any component of G and verify (3) to (9) by substitution. We note that although the presentation adopted here is the most economic one, we have originally calculated G by constructive methods : by functional derivation (as indicated in [8]) and also directly from the knowledge of the eigenfunctions of M .

To complete the job we spell out the solutions to the simple integral equations (17), (18). Both $\phi^\pm$ are continuous and can be expressed in terms of the solutions $C(\xi)$ and $S(\xi)$ of the hypergeometric equation $(1 - \xi^2)\ddot{f}(\xi) = 2f(\xi)$ belonging to the initial conditions $C(0) = 1$, $\dot{C}(0) = 0$ and $S(0) = 0$, $\dot{S}(0) = 1$, respectively.

$$\phi_k^+(u) = \begin{cases} q_0, & u < x_0 \\ A_k^+ S_u + B_k^+ C_u, & x_0 < u < k \\ a_k^+ \operatorname{sh}\left(\frac{u-k}{p}\right) + b_k^+ \operatorname{ch}\left(\frac{u-k}{p}\right), & k < u < x_1 \\ \phi_k^+(x_1), & x_1 < u \end{cases} \quad (21)$$

where $S_u = S\left(\frac{u}{\sqrt{k^2 + 4p^2}}\right)$ and similarly for C_u ,

$$A_k^+ = -q_0(\dot{C}_{x_0} - N_k C_{x_0})$$

$$B_k^+ = q_0(\dot{S}_{x_0} - N_k S_{x_0})$$

$$a_k^+ = q_0 \left\{ \frac{p}{\sqrt{k^2 + 4p^2}} [\dot{C}_k(\dot{S}_{x_0} - N_k S_{x_0}) - \dot{S}_k(\dot{C}_{x_0} - N_k C_{x_0})] + \right. \\ \left. + \frac{k}{2p} [C_k(\dot{S}_{x_0} - N_k S_{x_0}) - S_k(\dot{C}_{x_0} - N_k C_{x_0})] \right\} \quad (22)$$

$$b_k^+ = q_0 \{ C_k(\dot{S}_{x_0} - N_k S_{x_0}) - S_k(\dot{C}_{x_0} - N_k C_{x_0}) \}$$

and

$$N_k = \frac{\sqrt{k^2 + 4p^2} \cdot 4p^2 + k^2 + x_0^2}{x_0 \cdot 4p^2 + k^2 - x_0^2}$$

For $\phi_k^-(v)$ one has :

$$\phi_k^-(v) = \begin{cases} \phi_k^-(x_0), & v < x_0 \\ A_k^- S_v + B_k^- C_v, & x_0 < v < k \\ a_k^- \operatorname{sh}\left(\frac{x_1-v}{p}\right) + b_k^- \operatorname{ch}\left(\frac{x_1-v}{p}\right), & k < v < x_1 \\ 1, & x_1 < v \end{cases} \quad (23)$$

where

$$\begin{aligned}
 A_k^- = & -C_k \left\{ \frac{\sqrt{k^2 + 4p^2}}{p} \left[\operatorname{sh} \left(\frac{x_1 - k}{p} \right) + \frac{1 - x_1}{p} \operatorname{ch} \left(\frac{x_1 - k}{p} \right) \right] + \right. \\
 & + \frac{k}{2p^2} \sqrt{k^2 + 4p^2} \left[\operatorname{ch} \left(\frac{x_1 - k}{p} \right) + \frac{1 - x_1}{p} \operatorname{sh} \left(\frac{x_1 - k}{p} \right) \right] \left. \right\} - \\
 & - \dot{C}_k \left\{ \operatorname{ch} \left(\frac{x_1 - k}{p} \right) + \frac{1 - x_1}{p} \operatorname{sh} \left(\frac{x_1 - k}{p} \right) \right\} \\
 B_k^- = & S_k \left\{ \frac{\sqrt{k^2 + 4p^2}}{p} \left[\operatorname{sh} \left(\frac{x_1 - k}{p} \right) + \frac{1 - x_1}{p} \operatorname{ch} \left(\frac{x_1 - k}{p} \right) \right] + \right. \\
 & + \frac{k}{2p^2} \sqrt{k^2 + 4p^2} \left[\operatorname{ch} \left(\frac{x_1 - k}{p} \right) + \frac{1 - x_1}{p} \operatorname{sh} \left(\frac{x_1 - k}{p} \right) \right] \left. \right\} \\
 & + \dot{S}_k \left\{ \operatorname{ch} \left(\frac{x_1 - k}{p} \right) + \frac{1 - x_1}{p} \operatorname{sh} \left(\frac{x_1 - k}{p} \right) \right\}
 \end{aligned} \tag{24}$$

$$a_k^- = \frac{1 - x_1}{p}$$

$$b_k^- = 1.$$

Finally, for the Wronskian, one obtains

$$\begin{aligned}
 A_k = & \frac{2}{p} \left\{ a_k^+ \operatorname{ch} \left(\frac{x_1 - k}{p} \right) + b_k^+ \operatorname{sh} \left(\frac{x_1 - k}{p} \right) \right\} + \\
 & + 2 \frac{1 - x_1}{p^2} \left\{ a_k^+ \operatorname{sh} \left(\frac{x_1 - k}{p} \right) + b_k^+ \operatorname{ch} \left(\frac{x_1 - k}{p} \right) \right\}.
 \end{aligned} \tag{25}$$

With this we have compiled all the formulae that are needed for the various components of the Green function. Since the excitation spectrum consists of two bands, namely that of the large masses, with $m^2 \sim \tau$, and that of the small masses, with $m^2 \sim \tau^2$, there is no simple scaling in the Gaussian approximation. The behaviour of the first propagator and that of the trace of the resolvent in the various regions defined by these mass scales have been analysed to some extent in [8], where also the ghosts (i.e. the negative multiplicity modes) were identified as keeping the theory finite on the Gaussian level at the price of causing strong infrared singularities. We cannot go into similar details in here and restrict ourselves to a general statement only : the behaviour of all the off-diagonal components of G we have worked out so far is similar to that described in [8] for the diagonal ones.

It is well known that in the limit $x \rightarrow 1$ (or x_1) the order parameter represents the self overlap. A. P. Young has also suggested [12] that in the same limit (for all overlaps) the Green functions should be related to the behaviour in one single phase space valley. With our propagators this « single-valley » contribution to the correlation function

$$\overline{(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2} \text{ is } \lim_{x=x_1} \left(G_{1,1}^{x,x} - 2 G_{1,x}^{x,x} + \lim_{z=x_1} G_{z,z}^{x,x} \right) = \frac{1}{p^2}$$

a strikingly simple result.

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