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Random (free) energies in spin glasses

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Résumé. — Nous étudions le modèle d'énergies aléatoires généralisé de Derrida et la loi de probabilité qui en résulte pour les poids des niveaux d'énergie (libre). Nous montrons que celui-ci peut être utilisé pour représenter les énergies libres (dégénérées) du verre de spin dans la limite de champ moyen. Nous complétons ainsi la représentation proposée récemment par Mézard-Parisi-Virasoro.

Abstract. — We study the Derrida generalized random energy model and its resulting probability law for the weights of the (free) energy levels. We show that it can be used to represent the (degenerate) free energies of the mean field spin glass thus completing the picture proposed by Mézard-Parisi and Virasoro.

The Random Energy Model (REM) was introduced by Derrida [1] as a simplification of spin glass problems. The model consists of α_1^N independent random energy levels E_α drawn from a Gaussian probability law $\mathcal{P}_1(E) \sim \exp - (E^2/NJ^2 a_1)$ and carrying a weight $P_\alpha = \exp - \beta E_\alpha/Z$, $Z = \sum_\alpha \exp - \beta E_\alpha$. At some critical temperature $T_1 = (a_1/4 \ln \alpha_1)^{1/2}$ the system reaches zero entropy and « freezes ». Recently Derrida and Toulouse [2] were able to directly compute the moments of the probability law for the weights

$$W^{(k)} = \sum_\alpha \langle (P_\alpha)^k \rangle \quad (1)$$

where $\langle \rangle$ stands for average over the Gaussian distributions $\mathcal{P}_1$. They found that the $W^{(k)}$ were the same functions as those computed a while ago [3] (via replicas) for the Sherrington-Kirkpatrick (SK) model [4]. In the REM the argument of the W 's is T/T_1 , in the SK model $x(q_M)$, the edge of the upper plateau of the Parisi function, a correspondence that had been established by Gross and Mézard [5] starting from a spin representation [1] of the Derrida model.

Trying to take into account correlations between levels, Derrida has introduced recently [6] a Generalized Random Energy Model (GREM) where now the random levels are built as sum

of R independent random variables

$$E_\alpha \equiv E_{j_1 j_2 \dots j_R} = \varepsilon_{j_1} + \varepsilon_{j_1 j_2} + \dots + \varepsilon_{j_1 j_2 \dots j_R} \quad (2)$$

where $j_r = 1, 2, \dots, \alpha_r^N$. Each of the $\varepsilon_{j_1 j_2 \dots j_r}$ is drawn from a new Gaussian probability law

$$\mathcal{P}_r(\varepsilon) \sim \exp - (\varepsilon^2 / NJ^2 a_r). \quad (3)$$

The model exhibits R transitions at $T_r = (a_r / 4 \ln \alpha_r)^{1/2}$ (provided the parameters are such that $T_r < T_{r-1}$). Below T_r the tiers 1 through r are « frozen » in the sense that

(i) the thermal averages of $\varepsilon_{j_1}, \dots, \varepsilon_{j_1 \dots j_r}$ become temperature independent $E_1, \dots, E_r$ (with zero associated entropy),

(ii) the « efficacious » levels are no longer exponential in number, but a smaller number in the vicinity of $E_1, \dots, E_r$.

The hierarchical tree structure of the levels with R tiers thus provides a natural definition of the (co)distance r (or $x_r \equiv r/R$) between two levels whose energies have r identical components $\varepsilon_{j_1}, \varepsilon_{j_1 j_2}, \dots, \varepsilon_{j_1 j_2 \dots j_r}$ but a distinct $(r+1)$ th component. Still missing is a spin interpretation that (*via* a proof à la Gross-Mézard) would provide a direct definition of a monotonic overlap function q_r between those two levels.

Here we want to study the moments of the probability law for the weights $P_{j_1 j_2 \dots j_R}$ of levels having a given (co)distance x_r

$$W_r^{(k)} = \sum_{j_1 \dots j_r} \left\langle \left(\sum_{j_{r+1} \dots j_R} P_{j_1 \dots j_r j_{r+1} \dots j_R} \right)^k \right\rangle. \quad (4)$$

The first part of this paper provides an explicit form for $W_r^{(k)}$ at all temperatures.

Independently of the above developments of the GREM, Mézard, Parisi and Virasoro [7] have proposed a description of the standard mean field (SK) model that illuminates all the properties (e.g. sample to sample or valley to valley fluctuations) discovered in previous studies [3, 8]. They write the *free energy* of a state α as

$$F_\alpha = F_0 + f_\alpha \quad (5)$$

with F_0 the extensive, degenerate part and f_α a fluctuation of relative order $1/N$ and they assume that f_α is an independent random variable with an exponential probability law

$$Q(f) = \rho_M \exp \{ \rho_M (f - f^0) \} \theta(f^0 - f) \quad (6)$$

with $\rho_M = \beta x(q_M)$. They then show that all the moments $W^{(k)}$ (Eq. (1)) are identically those of the probability law for the weights P_α of reference [3]. Further defining the weight of a cluster [3] I_q characterized by the overlap q , $P_{I_q} = \sum_{\alpha \in I_q} P_\alpha$, they show that the same result holds provided

ρ_M is replaced by $\rho_q = \beta x(q)$.

They, however, do not say how one could define the free energy associated with a cluster and a naive but natural definition such as

$$\exp - \beta f_I = \sum_{\alpha \in I} \exp - \beta f_\alpha \quad (7)$$

leads to inconsistencies⁽¹⁾ as we shall see below. This has led us to propose to represent the free

⁽¹⁾ E.g. in dynamics [9] we are interested in computing $\langle \exp - tZ \rangle$. Then depending upon whether we choose to write Z as $\sum_\alpha$ or $\sum_{I_q}$ we get $\exp - (t) x(q_M)$ or $\exp - (t) x(q)$ respectively.

energies, along the lines of the GREM, as a sum of R components

$$F_\alpha \equiv F_{j_1 j_2 \dots j_R} = f_{j_1} + f_{j_1 j_2} + \dots + f_{j_1 j_2 \dots j_R}. \quad (8)$$

We show below that, provided we appropriately choose the parameters *viz.*

$$T/T_r = x(q_r) \quad (9)$$

we obtain a representation for the free energies of the SK model that reproduces all desired features (moments) and leaves *no ambiguity* for the definition of the cluster free energy $f^{(r)}$ (with codistance x_r , or overlap q_r) i.e.

$$\exp(-\beta f_{j_1 \dots j_r}^{(r)}) = \exp[-\beta(f_{j_1} + \dots + f_{j_1 \dots j_r})] \sum_{j_{r+1} \dots j_R} \exp[-\beta(f_{j_1 \dots j_{r+1}} + \dots + f_{j_1 \dots j_R})]. \quad (10)$$

This new proposal is the content of the last part of this paper.

1. On the moments $W_r^{(k)}$.

In the GREM the result for equation (4) reads

$$W_r^{(k)} = \Gamma(k - T/T_r)/\Gamma(k) \Gamma(1 - T/T_r) \quad T < T_r \\ = 0 \quad T > T_r \quad (11)$$

in agreement with the result of Derrida-Toulouse for $r = 1$. Starting from the paramagnet ($T > T_1$) and cooling down, one sees at T_1 the first tier freezing, i.e. $\tilde{\epsilon}_{j_1} = E_1 + \epsilon_{j_1}$ and the ϵ_{j_1} confined near E_1 . Correspondingly $W_1^{(k)}$ starts being non zero (some levels ϵ_{j_1} start to dominate). At T_2 the same phenomenon occurs with the second tier etc. Below T all $W^{(k)}$ are non-zero and « all tiers are frozen ».

As mentioned above there are two aspects in freezing :

(i) Frozen (free) energies E_r

$$E_r = -N(\alpha_r \ln \alpha_r)^{1/2} + (\alpha_r/16 \ln \alpha_r)^{1/2} \ln N \quad (12)$$

whose sum over r is the (free) energy E of the system.

(ii) Averages over α_r^N levels drawn from $\mathfrak{F}_r(\epsilon)$ (Eq. (3)) may be replaced by averages on M_r lowest levels, all located near E_r , the « efficacious levels », with a distribution (cf. Eq. (6))

$$\mathfrak{F}_r(\tilde{\epsilon}) = (1/T_r) \exp \{ (\tilde{\epsilon} - \epsilon_r^0)/T_r \} \theta(\epsilon_r^0 - \tilde{\epsilon}) \quad (13)$$

together with, for $M_r \rightarrow \infty$

$$M_r \exp - (\epsilon_r^0/T_r) = (4 \pi \ln \alpha_r)^{-1/2} \equiv V_r. \quad (14)$$

This result is implicit in Mézard-Parisi-Virasoro and is derived by replacing ϵ by $E_r + \tilde{\epsilon}$ in (3), using (12),

$$M_r \sim (\alpha_r)^N \int_{-\infty}^{E_r + \epsilon_r^0} \mathfrak{F}_r(\epsilon) d\epsilon \quad (15)$$

and expanding for large N .

(²) Neither for the evaluation [10] of $\langle \exp - tZ \rangle$.

2. Sketch of derivation, $R = 2$.

The tired reader may jump to section 3. We compute $W_2^{(2)} \equiv \sum_{j_1 j_2} \langle P_{j_1 j_2}^2 \rangle$

$$W_2^{(2)} = (\alpha_1 \alpha_2)^N \int_0^\infty d\lambda \lambda \left\{ \left[\int d\varepsilon_1 \mathcal{F}_1(\varepsilon_1) f_0^{\alpha_2^N}(\mu_1) \right]^{\alpha_1^N - 1} \times \right. \\ \left. \times \int d\varepsilon_1 \mathcal{F}_1(\varepsilon_1) e^{-2\beta\varepsilon_1} f_0^{\alpha_2^N - 1}(\mu_1) f_2(\mu_1) \right\} \quad (16)$$

$$f_k(\mu_1) = \int d\varepsilon_2 \mathcal{F}_2(\varepsilon_2) \exp - (\mu_1 e^{-\beta\varepsilon_2}) e^{-k\beta\varepsilon_2} \quad (17)$$

$$\mu_1 = \lambda e^{-\beta\varepsilon_1} \quad (18)$$

(i) $T < T_2 < T_1$: $\mathcal{F}_r$ ($r = 1, 2$) become exponential distributions $\tilde{\mathcal{F}}_r$ and $\alpha_r^N \rightarrow M_r$ as in equations (13) to (15). We get

$$f_2(\mu_1) \sim (T/T_r) (\mu_1)^{-2+(T/T_r)} \Gamma(2 - T/T_r) \quad (19)$$

also after partial integration

$$f_0^{\alpha_2^N}(\mu_1) \sim \exp \{ -v_r(\mu_1)^{T/T_r} \Gamma(1 - T/T_r) \}. \quad (20)$$

Those steps are identical to those taken in Mézard-Parisi-Virasoro. Except that here they are to be iterated. This is easy since the functions keep the same structure yielding

$$W_2^{(2)} = 1 - T/T_2. \quad (21)$$

(ii) $T_2 < T < T_1$: The unfrozen tier $r = 2$ keeps now the full Gaussian distribution,

$$f_k(\mu_1) = \pi^{-1/2} \int_{-\infty}^{+\infty} dy e^{-y^2} \exp - \{ \mu_1 e^{-\Lambda y} \} e^{-k\Lambda y} \quad (22)$$

with $\Lambda^2 = N\beta^2 J^2 a_2$. The large N expansion for f_k is given in reference [1]. But since $\mu_1 = \lambda \exp - \beta\varepsilon_1$, contains variables to be integrated upon, it is *a priori* not clear which is the important range of μ_1 as $N \rightarrow \infty$. We get

$$f_0^{\alpha_2^N}(\mu_1) = [1 - \mu_1 e^{\Lambda^2/4} + \dots]^{\alpha_2^N} \sim \exp - \{ \mu_1 x_2^N \} \quad (23)$$

and $x_2^N = \alpha_2^N \exp(\Lambda^2/4)$, the exponentiation in (23) being valid when $\mu_1 x_2^N \simeq \mathcal{O}(b^N)$, $b \leq 1$.

For $f_2(\mu_1)$ depending on the domain we get

$$e^{\Lambda^2} \quad \text{if} \quad \ln \mu_1 < -\Lambda^2 \\ [x_2^N \mu_1]^{-2-k(T)} [x_2^N]^{-k(T)/2} \quad \text{if} \quad -\Lambda^2 < \ln \mu_1 < -\frac{1}{2}\Lambda^2$$

here a temperature dependent factor is omitted and $k(T) = -(1/2) (1 + (T/T_r)^2)$.

For the ε_1 integration again $\alpha_1^N \rightarrow M_1$, $\mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1$ in (16). With any of the domains quoted above the integrals now reduce to types that already occurred in (i). The final λ integral is then seen to pick up its main contribution from $\lambda \sim \mathcal{O}(x_2^{-N} \exp \beta E_1)$ so that the condition to obtain (23) is

satisfied with $b = 1$ and consistency is assured. The result for $T_2 < T < T_1$ is then

$$W_2^{(2)} = 0. \quad (24)$$

(iii) $T_2 < T_1 < T$: In the paramagnetic region Gaussians have to be kept for both tiers, leading again to $W_2^{(2)} = 0$ but with a distinct approach to zero.

For arbitrary R a hierarchy of integrations to be carried in succession is involved. This complexity is compounded by the fact that, e.g. in $W_1^{(2)} = \sum_{j_1} \left(\sum_{j_2} P_{j_1 j_2} \right) \left(\sum_{j_2} P_{j_1 j_2} \right)$ one has to distinguish the case where $j_2 = j_2$ which for $W_r^{(k)}$ leads to a proliferation of distinct sectors.

3. On the definition of the cluster free energy.

Looking back on to the cluster free energy as defined in (7) together with probability law $Q(f_\alpha)$ (Eq. (6)) one may compute the corresponding law for f_i . In the limit $(f_1 - f^0) \rightarrow -\infty$ one obtains indeed an exponential distribution but with a decay constant $x(q_M)$ identical to that of f_α (instead of $x(q)$ as it would be naively expected). In contradistinction if we take a GREM with $R = 2$

$$E_{j_1 j_2} = \varepsilon_{j_1} + \varepsilon_{j_1 j_2} \quad (25)$$

and its effective (free) energy η_{j_1}

$$\exp - \beta(\varepsilon_{j_1} + \eta_{j_1}) = \exp - \beta \varepsilon_{j_1} \sum_{j_2} \exp - \beta \varepsilon_{j_1 j_2} \quad (26)$$

one finds that the η distribution is asymptotically exponential with a decay constant $1/T_2$ as above but the $(\varepsilon + \eta)$ distribution has the *smaller* decay constant $1/T_1$ (for $T_2 < T_1$) as we expected above.

4. The SK model as a GREM.

It is clear now that we may choose to represent the random free energies of the SK model (Eq. (5)) as a *sum* of independent random free energies (Eq. (8)). Provided we properly choose the parameters a_r, α_r . It suffices for that to recover

(i) the decay constant $x(q_r)$ *via*

$$T(4 \ln \alpha_r / a_r)^{1/2} \equiv T/T_r = x(q_r). \quad (27)$$

This insures that one has the proper probability law for all q 's derived by Mézard *et al.* [3].

(ii) the « frozen » values for the (extensive part of the) free energy

$$-NJ(a_r \ln \alpha_r)^{1/2} \equiv F_r = F_r^{\text{SK}}. \quad (28)$$

The SK free energy for the tier r may be conveniently written using Sompolinsky's representations [11, 12] in the Parisi gauge. Some degree of arbitrariness remains since only $\sum_r F_r^{\text{SK}} = F^{\text{SK}}$ is uniquely defined. Note that $F_r^{\text{SK}} \sim \dot{q}_r$, hence $F_r^{\text{SK}} = 0$ on the plateau of q_r .

Note finally that in the SK model *all* $W_r^{(k)}$ as given in (11) are *non zero*, and hence from this point of view *all tiers are frozen*, and all Gaussian become exponentials near their frozen values E_r . This expect on the plateau of q_r where the width of the Gaussian vanishes ($a_r \sim \dot{q}_r$) and the corresponding $f_{j_1 \dots j_r}$ components disappear. As $T \rightarrow T_c$ all components disappear to only leave the paramagnetic state.

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