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Stretched exponential relaxation in systems with random free energies

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Résumé. — Un verre de spin est caractérisé par un grand nombre d'états (ou fond de vallées) quasi-dégénérés. Il a été montré récemment que leurs énergies libres $F_a = F_0 + f_a$, où f_a est une fluctuation non extensive, se comportent en variables aléatoires *indépendantes*, comme dans le modèle de Derrida. Nous considérons la relaxation vers l'équilibre de tels systèmes. En particulier nous considérons une simple approximation pour la probabilité de transition (dans l'équation maîtresse) qui conduit à un temps de relaxation aléatoire de la forme $\tau \sim \exp \beta \Delta F$ où $-\beta(F_0 + \Delta F) = \ln \sum_a \exp(-\beta F_a)$.

La relaxation vers l'équilibre qui en découle, a la forme d'une *exponentielle étirée*, un comportement commun à une large classe de matériaux. La façon dont un tel résultat dépend du choix de la probabilité de transition, est abordée.

Abstract. — A spin glass phase is characterized by a large number of quasi degenerate states (or valley bottoms). Their free energies $F_a = F_0 + f_a$, where f_a is a non extensive fluctuation, have recently been shown to be random *independent* variables, as in Derrida's model. Relaxation to equilibrium of such systems is considered and a simple approximation to the transition probability (in the master equation) is considered that leads to a (random) relaxation time $\tau \sim \exp \beta \Delta F$ with $-\beta(F_0 + \Delta F) = \ln \sum_a \exp(-\beta F_a)$. The ensuing relaxation to equilibrium is shown to be a *stretched exponential* a behaviour common to a wide class of materials. The dependence of the result on the particular choice of the transition probability is touched upon.

Glassy relaxation, it has been pointed out [1-4], often follows a stretched exponential form and dynamical mechanisms have been suggested [5] that give rise to a $\exp - (t/\tau)^{1-n}$, $0 < n < 1$, relaxation. This is in particular the case of spin glass material below freezing temperature [6-7].

Concerning this last material, its mean field theory has recently been illuminated by the work of Mezard *et al.* [8]. They discovered that the free energies F_a of the degenerate states (valley bottoms) that characterize the long range Ising spin glass [9] below the freezing transition, are *independent* random variables. Conversely, starting from the random (free) energy model of

Derrida [10], Derrida and Toulouse [11] have independently shown that the equilibrium probability law for the state occupation $P_a^{\text{eq}} = \exp(-\beta F_a)/Z$, $\left(Z = \sum_a \exp -\beta F_a\right)$ is the same as the one derived a year ago [12] from the Parisi [13] mean field solution. The probability law found for the random variable F_a , or rather for the small (non extensive) fluctuation f_a

$$F_a \equiv F_0 + f_a \quad (1)$$

all F_a being quasi degenerate, is an exponential (as in Derrida's model, when his Gaussian probability law is linearized into an exponential around its frozen value).

In this short note we want to point to a very simple consequence of the above situation (independent random levels) for the dynamics of relaxation. Indeed what we show is that with detailed balance and the simplest possible choice for the transition probability occurring in the master equation (that governs the relaxation to equilibrium of $P_a(t)$) one obtains *stretched exponential relaxations* with $n = y(q_M) \equiv 1 - x(q_M)$. Here $q(x)$ is the Parisi order parameter, q_M its maximum value (Edwards-Anderson order parameter) and

$$y(q_M) = \overline{\sum_a P_a^2}. \quad (2)$$

This is for « self relaxation ». If one is looking for quantities involving state overlaps larger than $q(q < q_M)$ then $y(q_M)$ is replaced by $y(q)$ [8].

Let us thus write a master equation for the state occupation $P_a(t)$

$$\frac{\partial}{\partial t} P_a(t) = \sum_b W_{ab} P_b(t) - \sum_b W_{ba} P_a(t) \quad (3)$$

with the detailed balance condition

$$W_{ab} e^{-\beta f_b} = W_{ba} e^{-\beta f_a} \quad (4)$$

i.e.

$$W_{ab} = e^{-\frac{\beta}{2}(f_a - f_b)} M_{ab} \quad (5)$$

where M_{ab} is symmetric in a, b . The simplest choice that can be made that at the same time keeps long relaxation times, is $M_{ab}^0 = \tau_0^{-1} e^{-\frac{\beta}{2}(f_a + f_b)}$ i.e. W_{ab} being only dependent on the destination state a (other choices are considered below). If we call $F_0 + \Delta F$ the free energy associated with the sum over states

$$e^{-\beta(F_0 + \Delta F)} = \sum_a e^{-\beta F_a} \quad (6)$$

the above choice amounts to say that we have a single relaxation time τ ,

$$P_a(t) = P_a^{\text{eq}} + e^{-\frac{t}{\tau}(P_a(0) - P_a^{\text{eq}})} \quad (7)$$

with

$$\tau = \tau_0 e^{\beta \Delta F}. \quad (8)$$

This relaxation time depends on *all* the random free energies f_a introduced in (1) and can thus be considered as a stochastic variable governed by the probability law of Mezard *et al.*

$$\mathcal{P}(f_a) = \beta x \exp\{\beta x(f_a - f_0)\} \quad (9)$$

where f_0 is an upper cut-off that is allowed to become infinite with the number of states M with

$$M e^{-\beta x f_0} \rightarrow v_0 \quad (10)$$

v_0 a finite non universal constant and where we use the constant $x \equiv 1 - y$ (for a maximum overlap q_M when we limit ourselves to self relaxation).

Let us compute for example, using equations (7) and (9), the relaxation to its equilibrium value $y(q_M) \equiv 1 - x(q_M)$, of

$$1 - x(t) = \overline{\sum_a (P_a(t))^2} \quad (11)$$

where the bar stands for averaging over the probability law of the f_a 's. A direct and exact evaluation of equation (11) yields

$$x(t) - x = 2 \phi(\theta) - \phi(2^x \theta) - [e^{-2^x \theta}] \quad (12)$$

where x is the constant defined above and

$$\theta = v_0 \Gamma(1 - x) (t/\tau_0)^x \quad (13)$$

$$\phi(\theta) = \theta - \theta^{\frac{1}{x}} \Gamma\left(1 - \frac{1}{x}; \theta\right) \sim \frac{e^{-\theta}}{\theta} \frac{1 - x}{x}. \quad (14)$$

Here $\Gamma\left(1 - \frac{1}{x}; \theta\right)$ is the incomplete gamma function [14]. For large times, we get a stretched exponential behaviour

$$x(t) \simeq x + 2 \left(\frac{1 - x}{x} \right) e^{-v_0 \Gamma(1 - x) (t/\tau_0)^x} [v_0 \Gamma(1 - x) (t/\tau_0)^x]^{-1}. \quad (15)$$

Note that $x(q) \rightarrow 0$ as $T_c - T \rightarrow 0$ in the SK spin glass model, yielding a very slow relaxation (experiments [6, 7] show that the power of t in the exponent goes to zero as T approaches T_c). In contradistinction, $x(q) \rightarrow 1$ as $T_c - T \rightarrow 0$ for Derrida's model. In equation (12), the term in the square bracket is present for *pure* state initial conditions ($P_a(0) = \delta_{a;a_0}$) and absent for *homogeneous* initial conditions ($P_a(0) = M^{-1}$). Note the relationship

$$- [x(t)]_{\text{pure}} + [x(t)]_{\text{homog.}} = e^{-v_0 \Gamma(1 - x) (2t/\tau_0)^x}. \quad (16)$$

Likewise for equilibrium initial conditions associated with a shifted magnetic field (x' instead of x) one would get

$$- [x(t)]_{\text{pure}} + [x(t)]_{\delta h} = (1 - x') e^{-v_0 \Gamma(1 - x) (2t/\tau_0)^x}. \quad (17)$$

The stretched exponential behaviour seems rather ubiquitous (time correlation keep the same behaviour) and stable. For example, if we choose $\tau = \exp m\beta \Delta F(\beta)$ the exponent of t is changed from x to $x/[m(1 - x) + x]$. If we change M_{ab}^0 into $M_{ab}^0 \sum_c e^{-\lambda \beta f_c}$ then x becomes $x/[\lambda + 1 - x]$.

In both cases the stretched exponential structure is maintained. It would be gratifying to solve also for the case $M_{ab} = e^{-\lambda \beta (f_a + f_b)/2}$ ($\lambda = 1$ being the special case discussed above) and see whether the stretched exponential survives when $\lambda > 1$.

To summarize we have pointed out that independent random levels that occur in spin glasses, with a trivially simple hypothesis on the transition probability between levels (states) lead to the omnipresent [1-7] stretched exponential behaviour.

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Note added in proof:

The temperature dependence of the stretched exponential forms found in spin glass responses has been recently measured by R. Hoogerbeets, Wei-Li Luo and R. Orbach [15]. Their result is of the form $\exp - (t/\tau)^x$ with $\tau^{-1} \sim \exp - a/T$. If in (13) one recalls that the constant v_0 is given by (10) one recovers the same temperature dependence.

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