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Heteroclinic loops leading to hyperchaos

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Résumé. — On présente quelques formes d'orbites homoclines et de boucles hétéroclines comme mécanisme de formation d'hyperchaos, c'est-à-dire de comportement chaotique avec plusieurs exposants de Lyapounov positifs.

Abstract. — Some homoclinic orbits and heteroclinic loops are discussed as mechanisms leading to hyperchaos.

Apparently chaotic behaviour has been observed in numerical simulations of many non-linear ordinary differential equations but in almost all cases to date it has been impossible to prove the existence of strange attractors. The simplest mathematical models for which strange attractors are known to exist (Axiom A) have never been numerically observed in flows given by explicit ordinary differential equations. Lorenz-like attractors can be rigorously analysed [1] but occur far less frequently than the Henon-like attractors that are associated with the formation of horseshoes in systems with strong dissipation [2]. Unfortunately the existence of strange attractors has not been proven for such maps and it is possible that the invariant set consists of many small sinks. Nonetheless these attractors occur most frequently in situations of physical interest and can, in some sense, be thought of as strange.

A useful numerical measure of the complicated dynamics of flows is the spectrum of their Lyapounov exponents (see e.g. [3]). For attractors in $\mathbb{R}^3$ there can be at most one positive Lyapounov exponent. This is no longer true in higher dimensions. The cases where more than one Lyapounov exponent is positive are less well understood, and to distinguish these from the

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simpler forms of chaos we will use the term *hyperchaos* [4]. Understanding hyperchaos might be a step towards understanding the development of turbulence in systems governed by partial differential equations. In this Letter we propose mechanisms for creating hyperchaos in the same spirit as Shil'nikov's theorem [5] for creating horseshoes.

Figure 1 shows a generalization of the horseshoe to three-dimensional maps which can be thought of as return maps for four-dimensional flows. The asymptotic behaviour is a shift on $\Sigma_4 = \{1, 2, 3, 4\}^{\mathbb{Z}}$, i.e. the space of realizations of coin tossing with two distinguishable coins (recall that for the usual horseshoe only one coin is needed). Like the horseshoe, this map does not contain an attractor : most trajectories eventually leave the original cube (if the mapping is smooth enough [6]), but one observes chaos and hyperchaos in families of maps which evolve from a contracting map with a simple stable fixed point to a map like the one of figure 1 [4].

Fig. 1. — A generalized horseshoe : (a) take a cube, (b) fold in (x, z) , (c) fold in (x, y) . The four horizontal bars in (a) are mapped to the four vertical sheets inside the box in (c) so the symbolic dynamics of the mapping is given by a shift on Σ_4 .

Maps similar to that of figure 1 arise naturally in families of ordinary differential equations containing some sorts of heteroclinic loops. Here we consider only the simplest case in $\mathbb{R}^4$. All the arguments below can easily be extended to higher dimensions with more expanding directions.

The system shown in figure 2 has two fixed points A_0 and A_1 . Both A_0 and A_1 are saddle-foci with eigenvalues $(\lambda_j, -\rho_j \pm i\omega_j, -\nu_j)$ where $\nu_j > \lambda_j > \rho_j > 0$, $j = 0, 1$. There is a heteroclinic orbit, Γ_{01} , from A_0 to A_1 and another, Γ_{10} , from A_1 to A_0 , forming a heteroclinic loop. We shall see that such flows can induce a return map similar to the map of figure 1 on the small box Π_{00} near A_0 in figure 2.

The derivation of the return map T is standard [5, 7] : providing the vector field is C^{1+1} (i.e. differentiable, with first derivative Lipschitz continuous), it can be linearized by a smooth change of variables in a small neighbourhood of each of the stationary points A_j [8]. Then, following [5], one decomposes T as $T_{10} \circ T_{11} \circ T_{01} \circ T_{00}$. The meanings of $T_{00}, \dots, T_{11}$ are suggested in figure 2 and we refer the reader to [5] or [7] for details.

In appropriate local coordinates (x_i, y_i, z_i, w_i) $i = 0, 1$, the linearized equations near A_i are

$$\begin{aligned}\dot{x}_i &= -\rho_i x_i + \omega_i y_i \\ \dot{y}_i &= -\omega_i x_i - \rho_i y_i \\ \dot{z}_i &= \lambda_i z_i \\ \dot{w}_i &= -\nu_i w_i.\end{aligned}$$

$$\begin{aligned}x_0 &= c_{10} + B_1 x'_1 + C_1 y'_1 \\ y_0 &= 0 \\ z_0 &= a_1 w'_1 \\ w_0 &= c_{11} + D_1 x'_1 + E_1 y'_1\end{aligned}$$

where a_i and c_{ij} are non-zero constants and $\begin{pmatrix} B_i & C_i \\ D_i & E_i \end{pmatrix}$ are non-singular constant matrices, $i, j = 0, 1$. The effect of T_{ii} is to stretch the « box » Π_{ii} into a pyramidal shape and then roll it up about its tip. This gives a pseudo-cylinder P_i with its axis α_i in the w_i -direction in $\Pi_{ik}, k = i + 1 \pmod{2}$. The geometrical meaning of $T_{ij} (i = j)$ is, in general, to transport P_i to Π_{jj} . The particular choice we have made means that α_0 is mapped to the x_1 -axis by T_{01} and α_1 is mapped parallel to the z_0 -axis by T_{10} , so that the global effect of T after passing close to the two stationary points is to roll up the box Π_{00} successively in two orthogonal directions, and to put it back in Π_{00} as indicated in figure 3. In order to have hyperchaos we need an expanding direction in the plane of each of these two twists. This condition is satisfied providing $\frac{\rho_1}{\lambda_1} \frac{v_2}{\lambda_2} < 1$ and $\frac{\rho_2}{\lambda_1} \frac{v_1}{\lambda_2} < 1$ (cf. [5]).

Fig. 3. — The schematic image of Π_{00} under T as defined in the text. Cf. figure 1c.

This construction leaves us in exactly the same position vis-à-vis the generalized horseshoe of figure 1 as the map of Shil'nikov's theorem stand vis-à-vis the horseshoe. This is precisely the position we want to be in.

The heteroclinic loop described above is of codimension two (two control parameters are needed). One can obviously adapt these arguments to obtain flows in $\mathbb{R}^N$ with $N - 2$ positive Lyapounov exponents but these are of codimension $N - 2$.

The second mechanism we describe here involves a homoclinic orbit Γ which, as $t \rightarrow \pm \infty$, approaches the origin 0 where the linearized differential equation is (in polar coordinates (r_i, θ_i) , $i = 0, 1$)

$$\dot{r}_0 = -\rho r_0$$

$$\dot{\theta}_0 = \omega_0$$

$$\dot{r}_1 = \lambda r_1$$

$$\dot{\theta}_1 = \omega_1.$$

In order to describe the dynamics near Γ , we look at the return map T on a tube Π_0 on $r_0 = h_0 \ll 1$ with θ_0 near zero. Once again, we use the fact that there exists a smooth change of variables which linearizes the flow near 0 (providing the flow is C^{1+1} [8]). It is then natural to decompose T as $T_1 \circ T_0$. T_0 is a map from Π_0 to a tube Π_1 on $r_1 = h_1 \ll 1$ with θ_1 near zero. Assuming that

Π_0 and Π_1 are in the linearization domain, T_0 is given by

$$\begin{aligned} r'_0 &= h_0 \left(\frac{r_1}{h_1} \right)^{\rho/\lambda} \\ \theta'_0 &= \theta_0 + \frac{\omega_0}{\lambda} \ln \left(\frac{h_1}{r_1} \right) \\ r'_1 &= h_1 \\ \theta'_1 &= \theta_1 + \frac{\omega_1}{\lambda} \ln \left(\frac{h_1}{r_1} \right) \end{aligned}$$

T_1 represents the global reinjection from Π_1 to Π_0 (following Γ), and is thus a small perturbation of a linear map (with suitable change of coordinates, cf. [7]). Without doing further algebraic manipulations here, we show the action of T on Π_0 in figure 4. Like the map of figure 1, T contains horseshoes and, in particular, there is an infinite number of periodic orbits [9]. As has already been remarked in the usual situation [5], one cannot expect an attractor near Γ , but it is likely that attractors can be observed in nearby systems. Note that one needs $\rho < \lambda$ to generate periodic

Fig. 4. — This figure shows the effect of $T = T_1 \cdot T_0$ on a well-chosen shell Π_0 . The strips 1 and 2 are mapped back into Π_0 as shown (cf. the horizontal bars of figure 1a). The effect of T_0 is shown in stages : first stretch the radius, r_0 , then stretch θ_0 differentially with r_1 and twist θ_1 differentially.

orbits with two unstable directions with this mechanism, the flow is then volume expanding near 0, which is another reason why we cannot expect attractors near Γ .

There are other situations which may create hyperchaos which we have not mentioned : the simplest is when a system can be decomposed into the Cartesian product of two or more chaotic systems. This obviously produces hyperchaos in independent directions. As the product is broken this may persist for some range of parameter values.

In this Letter we have shown, using examples, that the study of homoclinic or heteroclinic trajectories can help in the understanding of hyperchaos. Hyperchaotic structures can occur in many distinct forms and we are not convinced of the need, or even the possibility, of a complete classification. It is our opinion, however, that when hyperchaos is observed in specific examples it can be analysed with the same methods as used here even if it does not correspond to one of our examples.

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