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Scenarios under constraint

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Résumé. — Nous étendons aux scénarios de transition vers le chaos, la notion de degrés de stabilité structurelle qui était déjà utilisée en théorie des bifurcations élémentaires. Nous donnons ensuite un exemple de scénario instable qui est stabilisé par une contrainte de symétrie.

Abstract. — We extend to the scenarios of transition to chaos, a notion of degree of structural stability, previously used for elementary bifurcations. We then give an example of unstable scenario which is stabilized by a symmetry constraint.

Given a dynamical system, one can define its degree of structural stability [1, 2], which is roughly the number of independent causes of structural instability.

For example, degree one structurally stable systems correspond to stable bifurcations, i.e. those bifurcations which, when they occur in a given continuous one parameter family of dynamical systems, also necessarily occur in any small perturbation of this family. These are in some sense the most interesting bifurcations, since they are the only ones which are likely to be observed in generic one parameter families.

It is natural to extend these notions to the scenarios of transition to chaos, and we shall focus here on these scenarios which can be analysed in term of a renormalization group (for short RG). To be consistent, it will be necessary to consider as distinct two scenarios which are topologically the same (e.g. the cascades of period doubling bifurcations in the families $1 - ax^2$ and $1 - bx^4$), but which have different universal properties. Then, the degree of structural stability of a scenario is one less than the number of unstable directions of the fixed point of the RG ⁽⁴⁾. Furthermore, the magnitudes of the corresponding eigenvalues allow a different kind of measure of the lack of structural stability in terms of the so-called crossover exponents.

An important remark about structurally unstable systems is that the degree of structural stability can be reduced by imposing appropriate constraints (e.g. symmetries, Hamiltonian character, ...).

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⁽⁴⁾ If the RG has a more complicated invariant set, one does not count the unstable directions corresponding to unstable manifolds contained in the invariant set.

In this Letter, we illustrate these ideas on a scenario of degree two of structural stability, which becomes stable when a symmetry constraint is imposed. The chosen example offers the advantage that not only the RG analysis can be achieved, but also one can describe completely the elementary bifurcations of the cascade. This scenario has been observed with explicit analytic ordinary differential equations [3] and partial differential equations [4]. For simplicity, we shall only consider it here in the case of geometrically defined differential equations in $\mathbb{R}^3$, which are linear in the parallelepiped $P : \{ (x, y, z) \mid -1 \leq x \leq 1, -\frac{1}{2} \leq y \leq \frac{1}{2}, 0 \leq z \leq 1 \}$, where they are given by :

$$\begin{aligned}\dot{x} &= \lambda_1 x \\ \dot{y} &= -\lambda_3 y \\ \dot{z} &= -\lambda_2 z \quad \text{with} \quad \lambda_3 > \lambda_2 > \lambda_1 > 0.\end{aligned}\tag{1}$$

We shall denote by U the upper face of P . The points of U with $x = 0$ have orbits which converge exponentially fast to 0. We shall suppose that from any other point of U , starts an orbit which leaves P and re-enters it by crossing U from above, so that all the interesting part of the dynamics is confined in the cell represented in figure 1. Such a flow induces a first return map F from $U \setminus \{x = 0\}$ to U and we shall only consider these flows such that F preserves the set of lines $\{x = \text{const.}\}$, i.e. F is given by :

$$F(x, y) = (f(x), g(x, y)),\tag{2}$$

where g acts as a contraction in the y direction, with $\frac{\partial g}{\partial y} \rightarrow 0$ as $x \rightarrow 0^\pm$.

Standard techniques allow then to describe the dynamics of F (and thus of the flow), in terms of the one-dimensional map f [5-7]. We shall only consider flows such that the orbit of $(1, 0, 0)$ (resp. $(-1, 0, 0)$) crosses U the first time at a point with a negative (resp. positive) x coordinate. Then, after suitable normalization, one can assume that f maps the interval $[-1, 1]$ into itself and is given by (up to higher order terms) [3] :

$$f_{\mu, \nu}(x) = f(x) = \begin{cases} 1 - \mu |x|^\alpha & \text{if } -1 \leq x < 0 \\ -1 + \nu |x|^\alpha & \text{if } 0 < x \leq 1 \end{cases} \quad \mu > 0, \nu > 0 \quad \text{and} \quad \alpha = \lambda_3/\lambda_1 \tag{3}$$

Fig. 1. — The cell which contains the non-trivial dynamics.

If the flow is invariant under the change of variables $(x, y, z) \rightarrow (-x, -y, z)$, we have $v = \mu$ in (3). Then, when μ varies, there is a cascade of symmetry breaking bifurcations, each of them being followed by a « gluing bifurcation » [8], where two cycles disappear to give a cycle which is close to the union of the two cycles which have disappeared. This has been shown in detail in [3], where it was also shown how the symmetry assumption allows one to reduce the study of f to the better understood problem of the iteration of the map $g(x) = 1 - \mu |x|^\alpha$. In order to study the scenario one can also write down directly the RG transformation for maps like (3). This renormalization transformation is defined by : (for a given $\alpha > 1$) :

$$\mathcal{R}f(x) = \frac{1}{\eta} f(\eta x) \quad \text{with} \quad \eta = \psi(1) \quad (4)$$

ψ being the fixed point of $\mathcal{R}$ with $\psi(x) = \psi(|x|^\alpha)$. If we now recall that the RG transformation for continuous maps of the interval is given by :

$$\mathcal{N}g(x) = -\frac{1}{\lambda} g(g(\lambda x)) \quad \text{with} \quad \lambda = -\varphi(1), \quad (5)$$

(φ being the fixed point of $\mathcal{N}$ with $\varphi(x) = \Phi(|x|^\alpha)$), it is straightforward to verify that $\psi(x) = -\text{sgn}(x) \varphi(x)$. As long as we are considering the symmetric case, only odd perturbations of the fixed point have to be taken into account in the stability analysis of $\psi(x)$, namely :

$$e(x) = \text{sgn}(x) h(x) \quad \text{where} \quad h(x) \equiv H(|x|^\alpha), \quad (6)$$

and the eigenvalue equation $D\mathcal{R}_{\psi(x)} e(x) = \delta e(x)$ reduces to :

$$-\frac{1}{\lambda} [h(\varphi(\lambda x)) + \varphi'(\varphi(\lambda x)) h(\lambda x)] = \delta h(x). \quad (7)$$

One recognizes in (7) the eigenvalue equation $D\mathcal{N}_{\varphi(x)} h(x) = \delta h(x)$ of the cascade of period doubling bifurcations. For α close to 1 ⁽⁵⁾ one gets $\delta = 2 + \mathcal{O}\left(\frac{1}{|\log \lambda|}\right)$ as the unique relevant eigenvalue out of the unit circle [9]. Hence the scenario is stable and the value of δ corresponds to the accumulation rate of the cascade. If we now release the symmetry constraint one has to consider also perturbations of the form $e(x) = k(x) = K(|x|^\alpha)$ which leads to the eigenvalue equation

$$\frac{1}{\lambda} [k(\varphi(\lambda x)) - \varphi'(\varphi(\lambda x)) k(\lambda x)] = \gamma k(x). \quad (8)$$

A computation similar to the one done in [9] for δ shows that (8) gives a new relevant eigenvalue $\gamma = \frac{2}{\lambda} + \mathcal{O}(1)$ for α close to one ⁽⁵⁾. This shows that this cascade is unstable with degree two of structural stability for close to one ⁽⁶⁾ (numerical simulations and [8] suggest that this remains true for any $\alpha > 1$). Crossover behaviour are then deduced as usual. For instance the topological

⁽⁵⁾ We restrict ourself here to the case of α close to one because it allows perturbation calculations. Note that for the kind of flows we consider $\alpha = \lambda_3/\lambda_1$ can be any real number larger than one. Also, because of the nature of the problem only fixed points of $\mathcal{R}$ and eigenvectors analytic in $|x|^\alpha$ have to be considered.

⁽⁶⁾ The same is true for the usual cascade of period doubling if, beside perturbations of the form $F(|x|^\alpha)$, one also considers the physically non admissible perturbations $\text{sgn} |x| G(|x|^\alpha)$. This instability yields to the cycle 2 of the RG reported in [10].

entropy near the critical point $\nu = \mu = \mu_c$, behaves like

$$\begin{aligned} h(f_{\mu,\nu}) &\sim (\mu - \mu_c)^{\frac{\log \delta}{\log 2}} H\left(\frac{(\mu - \nu)}{(\mu - \mu_c)^\Delta}\right), \quad \text{for } \mu > \mu_c, \\ h(f_{\mu,\nu}) &\sim |\mu - \nu|^{\frac{\log \gamma}{\log 2}} \tilde{H}\left(\frac{(\mu - \mu_c)}{|\nu|^{1/\Delta}}\right), \quad \text{for } \mu < \mu_c, \end{aligned} \quad (9)$$

where H and $\tilde{H}$ are universal functions and $\Delta = \frac{\log \delta}{\log \gamma}$ is the so-called crossover exponent.

The instability of this scenario is related to the instability of both kinds of elementary bifurcations which constitute the cascade. In both cases, one has a degree two of structural stability. If one follows a one parameter family of almost symmetrical flows, the pitchfork will be deformed to a saddle-node followed by an exchange of stability (see e.g. Ref. [11]). The unfolding of the gluing bifurcation is more complicated : in the parameter region between the existence of two stable cycles with period 2^n and the existence of a stable 2^{n+1} -cycle, one will cross infinitely many small intervals of the parameter values where one has a stable periodic orbit with period $2^n \cdot p$, separated by parameter values where one has an asymptotically stable aperiodic orbit. This has been illustrated in figure 2.

There are other known examples where a neat bifurcation, when perturbed, is replaced by a fuzzy one with the same initial and final states, separated by very complicated other dynamical states. In cascades of such bifurcations, it is the overlap of these fuzzy regions which will cause the instability of the scenario.

An example of such instability is provided by adding noise to the usual period doubling cascade [11]. In this case, the constraint which stabilizes the scenario corresponds trivially to suppressing the noise. Table I summarizes the stability property of some of the known renormalizable scenarios for smooth and dissipative deterministic dynamical systems (the effect of noise has not been considered). These results have been announced in [12].

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Fig. 2. — (a) Deformation of the pitchfork in our model ; (b) unfolding of the gluing, as seen in the flow for the simplest cycles. In the two cusped regions, we have suggested the intricate succession of an infinity of different dynamical behaviour.

Table I. — *Stability of scenarios, stabilizing constraints and experimental observations.*

Degree of structural stability	Scenarios	Example of stability constraint	Experimental observation at this time
$d = 1$	<ul style="list-style-type: none">— generic period doubling (quadratic fixed point)— intermittancies— Newhouse-Palis-Takens transition	stable scenarios	<div>yes</div> <div>yes</div> <div>yes</div>
$d = 2$	<ul style="list-style-type: none">— transition through quasi-periodicity— period doubling with quartic fixed point— the example in the main text	<div>follow a given rotation number</div> <div>no simple specification in physical experiments</div> <div>impose an appropriate symmetry</div>	<div>no</div> <div>no</div> <div>no</div>
$d > 2$	<ul style="list-style-type: none">— period doubling with higher order fixed point— cascades of doublings for tori (*)— intermitting for tori (*) <div>(*) $d > 2$ (indeed $d = \infty$), at this writing, established only for tori which support motions with rationally related frequencies.</div>	<div>no simple specification in physical experiments</div> <div>trivial decoupling of oscillators</div> <div>trivial decoupling of oscillators</div>	<div>no</div> <div>no</div> <div>yes</div>

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