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Accurate critical exponents from the ε -expansion

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Résumé. — Les exposants critiques ont été maintenant calculés dans le développement en ε jusqu'à l'ordre ε^5 . Nous avons appliqué à ces développements une méthode de sommation basée sur une transformation de Borel et une transformation conforme. Nous avons ainsi obtenu de nouvelles évaluations des exposants critiques, tout à fait cohérentes en deux dimensions avec les valeurs exactement connues du modèle d'Ising, et en trois dimensions avec les évaluations précédemment obtenues à partir du développement perturbatif à dimension fixée.

Abstract. — The ε -expansion of the critical exponents for the N -vector model is now available up to order ε^5 . Using a summation method based on a Borel transformation and a mapping, we obtain from these ε -expansions new estimates for the critical exponents, quite consistent in two dimensions with the exact values of the Ising model and in three dimensions with the estimates previously obtained from the perturbative expansion at fixed dimension.

1. Introduction.

Values of the critical exponents of the N -vector model are among the most successful quantitative predictions [1, 2] of the Renormalization Group approach [3-5] to critical phenomena.

The most accurate estimates of these exponents have been obtained up to now in the field theoretical formulation of the Renormalization Group, involving the $g(\phi^2)^2$ field theory, where ϕ is an N -component vector field.

The now « standard » Renormalization Group estimates [1] of these exponents have been calculated in this field theoretical approach by a summation of their divergent expansion in the renormalized coupling constant g , at fixed dimension $d = 3$ [6], these expansions being known up to order g^6 .

On the other hand, the first field theoretical estimates of the critical exponents came from the famous $\varepsilon = 4 - d$ expansion of Wilson and Fisher [4]. Unfortunately, until recently the ε -series [7, 8] were too short to lead to accurate estimates [1]. However, the ε -expansion is now known up to order ε^5 [9, 10].

We present here results given by the summation of these divergent series in ε . The method we use, based on a Borel transformation and a mapping, is similar to the one we used for the perturbation series in g at fixed dimension, but with a slight modification which will be explained below.

The main interest in this calculating is the following.

By setting $\varepsilon = 1$ ($d = 3$) we get a consistency check for the standard Renormalization Group estimates [1] of the critical exponents.

On the other hand, by setting $\varepsilon = 2$ ($d = 2$) and $N = 1$, we can compare the Renormalization Group predictions with the exactly known critical exponents of the two-dimensional Ising model. Up to now the only direct comparison came from the perturbation series in g at fixed dimension two, which are only known up to order g^4 and lead to rather unaccurate estimates, the comparison being thus not really meaningful [1].

As is shown in tables I, II and III, the situation is now extremely satisfactory.

2. The summation method.

In reference [10] the ε -expansion at order ε^5 has been summed by using the Borel mapping method in the form it has been applied to the series at fixed dimension. The results were not always accurate. In particular, the authors concluded that a sensible comparison between the exponent $\eta = 0.25$ of the two-dimensional Ising model and the corresponding estimate obtained from the ε -expansion, would require the calculation of at least two additional terms in the ε -expansion of η .

In this Letter, we show how a slight modification of the method allows us to obtain, on the contrary, good estimates for all critical exponents from their ε -series at order 5 : the comparison of our results for $\varepsilon = 2$, $N = 1$ with the two-dimensional Ising model is quite successful, even for the exponent η ; and our predictions for $\varepsilon = 1$ are in very good agreement with the standard three-dimensional Renormalization Group estimates obtained from the g -series.

We start, for any critical exponent $E(\varepsilon)$, from its ε -expansion :

$$E(\varepsilon) = \sum_k E_k \varepsilon^k \quad (1)$$

for which the large-order behaviour is known [11]

$$E_k \underset{k \rightarrow \infty}{\sim} k! a^k k^{b_0} c \quad (2)$$

with $a = -3/(N + 8)$ and with, for example, $b_0 = 3 + N/2$ for η and $b_0 = 4 + N/2$ for $1/\nu$, and where the c 's have been recently calculated [12].

As we have explained elsewhere [1, 13], it is natural in such a situation to use a Borel transformation :

$$E(\varepsilon) = \int_0^\infty t^b e^{-t} B(\varepsilon t) dt \quad (3)$$

$$B(\varepsilon t) = \sum_k \frac{E_k (\varepsilon t)^k}{\Gamma(k + b + 1)} \quad (4)$$

From the large-order behaviour, we know that the singularity of B closest to the origin is located at the point $-1/a$, and thus we map the cut-plane onto a circle of radius one, leaving the origin invariant, by :

$$\varepsilon t = \frac{4}{a} \frac{U}{(1 - U)^2} \quad (5)$$

Table I. — *Estimates of two-dimensional critical exponents from the ε -expansion at order ε^5 by the method of this Letter.*

	γ	ν	β	η	ω
$N = 0$	1.39 ± 0.04	0.76 ± 0.03	0.065 ± 0.015	0.21 ± 0.05	1.7 ± 0.2
$N = 1$	1.73 ± 0.06	0.99 ± 0.04	0.120 ± 0.015	0.26 ± 0.05	1.6 ± 0.2

Table II. — *Estimates of three-dimensional critical exponents from the ε -expansion at order ε^5 by the method of this Letter.*

	γ	ν	β	η	ω
$N = 0$	1.160 ± 0.004	0.5885 ± 0.0025	0.3025 ± 0.0025	0.031 ± 0.003	0.82 ± 0.04
$N = 1$	1.239 ± 0.004	0.6305 ± 0.0025	0.3265 ± 0.0025	0.037 ± 0.003	0.81 ± 0.04
$N = 2$	1.315 ± 0.007	0.671 ± 0.005	0.3485 ± 0.0035	0.040 ± 0.003	0.80 ± 0.04
$N = 3$	1.390 ± 0.010	0.710 ± 0.007	0.368 ± 0.004	0.040 ± 0.003	0.79 ± 0.04

Table III. — *Standard Renormalization Group estimates [1] of the three-dimensional critical exponents.*

	γ	ν	β	η	ω
$N = 0$	1.1615 ± 0.0020	0.5880 ± 0.0015	0.3020 ± 0.0015	0.027 ± 0.004	0.80 ± 0.04
$N = 1$	1.2410 ± 0.0020	0.6300 ± 0.0015	0.3250 ± 0.0015	0.031 ± 0.004	0.79 ± 0.03
$N = 2$	1.3160 ± 0.0025	0.6690 ± 0.0020	0.3455 ± 0.0020	0.033 ± 0.004	0.78 ± 0.025
$N = 3$	1.386 ± 0.004	0.705 ± 0.003	0.3645 ± 0.0025	0.033 ± 0.004	0.78 ± 0.02

Using the convergent expansion :

$$(1 - U)^\alpha B(\varepsilon t) = \sum_k A_k(b, \alpha) U^k, \tag{6}$$

one obtains for $E(\varepsilon)$ the expansion :

$$E(\varepsilon) = \sum_k A_k(b, \alpha) \int_0^\infty \frac{t^b e^{-t} U^k}{(1 - U)^\alpha} dt. \tag{7}$$

As explained in detail in reference [1], b and α are used as variational parameters to decrease the influence of the singularities of $B(\varepsilon t)$ at and close to $-1/a$, and at large distance in the complex plane.

The modification that we introduce in the present case, and which improves markedly the results, is the following : we know that, as functions of ε , the exponents $E(\varepsilon)$ for $N = 2$ and 3 have a singularity at $\varepsilon = 2$ ($d = 2$). For $N = 1$ a singularity probably lies at $\varepsilon = 3$ ($d = 1$), and for $N = 0$ presumably at $\varepsilon = 4$ ($d = 0$). Therefore the Borel transform B has an exponential behaviour for large values of the argument. To improve the situation, we have made a homographic transformation :

$$\varepsilon' = \rho\varepsilon/(\rho - \varepsilon). \quad (8)$$

We thus generate from (1) a new expansion of E in powers of ε' (with a similar large order behaviour), which we sum with the method described above (Eqs. (3)-(7)). If E would have no other singularities in ε , the best choice would be $\rho = \rho_0$ with $\rho_0 = 4$ for $N = 0$, $\rho_0 = 3$ for $N = 1$ and $\rho_0 = 2$ for $N \geq 2$. But we have some numerical evidence for other complex singularities, and we have therefore kept ρ as a variational parameter to minimize the exponential growth of the Borel transform in the new variable, which could then be handled more effectively by the parameter α .

As a consequence, we construct for $E(\varepsilon')$ an expansion analogous to that of equation (7), function of three variational parameters ρ , b and α , adjusted [1] to yield the best apparent convergence of the results when the order varies from 1 to 5.

3. Results.

The estimates we finally obtain for the various critical exponents are given in table I for $d = 2$ ($N = 0, 1$) and in table II for $d = 3$ ($N = 0, 1, 2, 3$). The critical exponents given in these tables have all been calculated independently from their ε -expansion. Taking, for instance, γ and ν as the two independent critical exponents, one can compare the estimates of β and η either by the direct calculations, or through the scaling relations $\beta = (d\nu - \gamma)/2$ and $\eta = 2 - \gamma/\nu$. One can easily check that the scaling relations between exponents are well verified within the quoted errors. This is a check of our resummation method, since the scaling relations hold order by order in perturbation theory.

Table I shows the remarkable agreement we obtain for $d = 2$, $N = 1$ with the exact values of the critical exponents of the two-dimensional Ising model :

$$\gamma = 1.75, \quad \nu = 1. \quad , \quad \beta = 0.125, \quad \eta = 0.25.$$

This reinforces our confidence in our summation method and provides, if still needed, an additional indication that the φ_2^4 field theory from the point of view of critical properties belongs to the same universality class as the two-dimensional Ising model. On the other hand, our result for ω is compatible with the recent result $\omega\nu = 1.35 \pm 0.25$ obtained from high temperature series [14].

For $d = 2$, $N = 0$ let us recall the recent Nienhuis' conjectures [15] $\gamma = 43/32 \simeq 1.344$, $\nu = 0.75$, $\beta = 5/64 \simeq 0.078$ and $\eta = 5/24 \simeq 0.21$. While quite compatible for ν and η , these conjectures are for γ and β only marginally compatible with our results.

Finally, table II shows for $d = 3$ the remarkable consistency between our results from the ε -expansion and the standard Renormalization Group estimates [1] for the critical exponents recalled in table III, keeping in mind that the known series have one order less in the ε -expansion than in the g -expansion.

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