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A simple model describing $^3$He-$^4$He mixtures near a wall

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Abstract. — We describe the superfluid ordering taking place near a wall in $^3$He-$^4$He mixtures by means of a semi-infinite version of the Blume-Emery-Griffiths model. A renormalization group analysis of the simplest version of this model yields a phase diagram which compares satisfactorily with available experimental data.

The phenomenon of superfluid film formation near a wall in $^3$He-$^4$He mixtures has been observed for many years [1]. It originates in an effective repulsion of $^3$He atoms by the wall. The van der Waals interactions between the wall and $^3$He or $^4$He atoms are equal, but $^3$He atoms occupy a larger volume because of their larger zero-point motion. One observes therefore a higher $^4$He concentration near the wall, which may induce a local superfluid ordering [2].

This transition occurs on a line which branches from the $\lambda$-line and extends to higher $^3$He concentrations, where one eventually observes phase separation near the wall. Figure 1 shows the bulk $^3$He-$^4$He diagram, with the available data on the location of these transitions. Although various experimental methods have been used to achieve this localization [1], little is known on the details of the transition. In particular, the thickness of the film, the order of the transition, the parameter dependence of the phase diagram have not yet been systematically investigated.

In a previous paper [3] (hereafter referred to as I) we pointed out that the superfluid film formation near the $\lambda$-line is analogous to the surface and special transitions in semi-infinite magnetic systems [4]. We described this phenomenon by means of a simple semi-infinite model with short-
range interactions, based on the Blume-Emery-Griffiths [5] (BEG) model for $^3$He-$^4$He mixtures. We conjectured that the special transition in this model belonged to the same universality class as the « usual », semi-infinite Ising models. Our arguments relied upon a mean-field calculation of the behaviour of the semi-infinite BEG model near the special transition. We also suggested that the same model could be used to describe qualitatively the behaviour of $^3$He-$^4$He mixtures in the whole range of $^3$He concentrations.

In this paper we confirm all the conjectures made in I. The simplest version of the semi-infinite BEG model has been applied to the description of the whole range of $^3$He concentrations.

A renormalization group analysis of its behaviour (within the Migdal-Kadanoff approach) has confirmed the stability of the fixed point describing the special transition in the pure Ising model against the perturbation due to a non vanishing concentration of impurities. We have obtained a phase diagram which compares surprisingly well with the existing experimental data.

1. The semi-infinite BEG model.

The BEG model, introduced [5] to describe $^3$He-$^4$He mixtures, is defined as follows. To each point $i$ of a simple cubic lattice are associated two variables: $t_i$, $\sigma_i$. The variable $t_i$ takes the values 0 (corresponding to the presence of a $^3$He atom) or 1 ($^4$He atom). The variable $\sigma_i$ takes the values $+1$ or $-1$ and represents the phase of the $^4$He wavefunction. This a drastic simplification, since one neglects the continuous $U(1)$ symmetry of the superfluid order parameter. This is all the more important in surface phenomena, due to the peculiar nature of superfluid ordering in two dimensions [7] (we will return to this point in the conclusions).

The partition function of the model is given by:

$$Z = \frac{1}{2^N} \sum_{(t,\sigma)} e^{-\beta H},$$

where $N$ is the total number of lattice points, the sum runs over all configurations of the variables $t_i$, $\sigma_i$ and

$$-\beta H = - \sum_{\langle ij \rangle} J_{ij} t_i t_j (\sigma_i - \sigma_j)^2 - \sum_{\langle ij \rangle} K_{ij} (t_i - t_j)^2 + \sum_i \Delta_i t_i. $$

![Phase diagram for $^3$He-$^4$He mixtures. Experimental data for surface transitions, taken from Romagnan et al. [1], were obtained by several research groups.](image)
The last sum runs over all points in the lattice; the first two ones over all nearest-neighbour pairs. The first term represents the ordering interaction among $^4$He atoms; the second describes « isotope » effects (i.e. the difference between the interactions in the $^3$He-$^3$He and $^4$He-$^4$He pairs); the parameter $\Delta$ plays the role of a chemical potential for $^4$He atoms.

This model describes well the observed phase diagram of bulk $^3$He-$^4$He mixtures already at the mean-field level, yielding superfluid ordering in homogeneous mixtures and phase separation of sufficiently low temperatures [5]. One can straightforwardly extend this model to semi-infinite geometries by taking:

$$ J_{ij} K_{ij} = \begin{cases} J, K & \text{if } i \text{ or } j \text{ belong to the bulk} \\ J_0, K_0 & \text{if } i \text{ and } j \text{ belong to the surface} \end{cases} $$

$$ \Delta_i = \begin{cases} \Delta & \text{if } i \text{ belongs to the bulk} \\ \Delta & \text{if } i \text{ belongs to the surface} \end{cases} $$

Thus, even such a simple model is already described by six parameters. In the absence of more detailed experimental data, which would allow to assign values to some of them, we have preferred to analyse the simplest version. We take therefore $K_{ij} = J_{ij}$ which corresponds to the Blume-Capel [8] limit of the BEG model. This amounts to neglecting the differences in the interactions among different isotopes and is quite justified experimentally [5]. We also set $J_0 = J$, supposing no direct effect of the wall on interatomic interactions. We finally take

$$ \Delta_0 = \Delta + \phi/kT $$(5)

where the parameter $\phi$ reflects the phenomenon of effective $^3$He repulsion by the wall. As shown in I, the line of onset of film superfluid joins the (bulk) $\lambda$-line at a point whose position depends on the value of $\phi$. All the data available to us agree in placing this point at a $^3$He concentration $x_3 \simeq 0.54$. (This agreement is probably due to the formation of a solid helium layer on the wall of the container, what makes almost irrelevant the composition of the wall.)

2. Renormalization group analysis.

We have performed a real-space renormalization group investigation of this model. The recursion relations which correspond to the doubling of the lattice constant have been derived by Kadanoff's [6] bond moving approach. The one-body interactions have been evenly distributed among the bonds which end at any given lattice point and are then moved along with them [9]. One thus obtains the following recursion relations for the bulk parameters:

$$ e^{-4J'} = (e^{-2\bar{K}} + e^{3-4\bar{J}})/(e^{-2\bar{K}} + e^{3-4\bar{J}} \text{ch } 4\bar{J}) $$

$$ e^{-2K'} = e^{-2\bar{K}}(1 + e^{3-2\bar{J}} \text{ch } 2\bar{J})^2/[(e^{-2\bar{K}} + e^{3-4\bar{J}} \text{ch } 4\bar{J})(1 + e^{3-2\bar{K}})] $$

$$ e^{4\phi'} = e^{4\Delta'}[(e^{-2\bar{K}} + e^{3-4\bar{J}} \text{ch } 4\bar{J})(1 + e^{3-2\bar{K}})]^3 $$

where

$$ \bar{J} = 4J; \quad \bar{K} = 4K; \quad \bar{\Delta} = 4\Delta/3. $$
The recursion relations for the surface parameters $J_0$, $K_0$, $\Delta_0$ have the following expressions:

\[ e^{-4J_0} = \left( e^{-2\tilde{K}_0} + e^{2\tilde{J}_0 - 4\tilde{J}_0} \right) \left( e^{-2\tilde{K}_0} + e^{2\tilde{J}_0 - 4\tilde{J}_0} \cosh 4\tilde{J}_0 \right), \]
\[ e^{-2K_0} = e^{-2\tilde{K}_0} \left( 1 + e^{2\tilde{J}_0 - 2\tilde{J}_0} \cosh 2\tilde{J}_0 \right)^2 \left( e^{-2\tilde{K}_0} + e^{2\tilde{J}_0 - 4\tilde{J}_0} \cosh 4\tilde{J}_0 \right) \left( 1 + e^{2\tilde{J}_0 - 2\tilde{K}_0} \right) \]
\[ e^{4\tilde{J}_0} = e^{12\Delta_0/5 + 2\Delta/3} \left[ (e^{-2\tilde{K}_0} + e^{2\tilde{J}_0 - 4\tilde{J}_0} \cosh 4\tilde{J}_0) \left( 1 + e^{2\tilde{J}_0 - 2\tilde{K}_0} \right) \right]^2 \times \left[ (e^{-2\tilde{K}_0} + e^{2\tilde{J}_0 - 4\tilde{J}_0} \cosh 4\tilde{J}_0) \left( 1 + e^{2\tilde{J}_0 - 2\tilde{K}_0} \right) \right]^{1/2} \]

where
\[ \tilde{J}_0 = 2J_0 + J, \quad \tilde{K}_0 = 2K_0 + K, \quad \tilde{\Delta}_0 = 4\Delta_0/5 + \Delta/3. \]

These recursion relations yield a very complicated flow pattern in the six-dimensional parameter space. (Already for the bulk equations (6)-(9) one has no less than thirteen fixed points [10].) The Blume-Capel limit and the $J_0 = J$ relations are not conserved by the flow. Since we are mainly interested in the phase diagram we use the equations just as a tool for computing the free energy. We iterate them until we reach a fixed point. The nature of the fixed point yields information about the phase we are in. We may thus identify the phase transition lines. The method of Nauenberg and Nienhuis [11] allows us to compute the free energy, and therefore, by numerical derivation, the $^3$He concentration $x_3$.

Figure 2 shows the results of this computation. The temperature scale is fixed by setting the tricritical temperature at its experimental value. The value of the « rejection parameter » $\phi$ is chosen in such a way that the special transition takes place at $x_3 \approx 0.54$. The phase diagram we obtain has quite similar features to the experimental one. Two interesting features are not apparent in this diagram:

(a) near the special transition, the surface transition line is curved, according to the results of a scaling analysis (see I, Eq. (7)), however this takes place in an exceedingly small temperature and concentration range ($|\Delta T| \ll 0.05$ K);

(b) at concentrations $x_3$ between 0.85 and 0.95 some changes in the regimes of the stability of the surface fixed points have been observed. This is also the region where the surface phase

![Fig. 2. — Phase diagram for $^3$He-$^4$He mixture obtained from the simplest, semi-infinite BEG model. Temperature scale is fixed by the tricritical point temperature ($T_T \approx 0.87$ K). Solid lines correspond to continuous transitions, dotted lines correspond to first order transitions (bulk and surface).](image)
transition changes from second to first order. The small value of the surface $^3$He concentration in this region makes a detailed analysis difficult. It is possible that in a certain range of temperatures one first observes, in decreasing $x_3$, a phase separation with appearance of a film rich in $^4$He, but not superfluid near the wall. This film then becomes superfluid at a lower value of $x_3$. The line of phase separation would end in a critical point. This behaviour would not be surprising, since it appears in a large region of parameter space in the two-dimensional BEG model, as shown e.g. by Berker and Wortis [10].

In fact the Nice experimental group reported in 1978 [1] this possibility of surface phase separation occurring prior to the appearance of superfluidity in the region of high $^3$He concentrations. These experiments are rather incomplete and one needs fairly detailed and thorough measurements both of the $^4$He concentration and of the superfluid order parameters to verify whether this unusual phenomenon really takes place. We think however that the results of Romagnan et al. [1] support our calculations.

3. Discussion.

This investigation was motivated by the need of proving that a simple statistical model is able to describe the essentials of the phenomena observed near a wall in $^3$He-$^4$He mixtures. We have seen that even a very simple Ising-like model is sufficient, if the semi-infinite geometry is taken into account and the microscopic « rejection effect » near the wall is represented by a single short-range chemical potential term.

In order to obtain a more quantitative description of the phenomena, a more realistic model is needed. The continuous nature of the symmetry broken in the phase transition should be first of all taken into account [12]. This may be obtained by use of the vector generalization of the BEG model [13]. Berker and Nelson have shown that the tricritical point is absent in such a model in two dimensions, and is replaced by a critical endpoint on the phase separation line. One may therefore argue that the phase diagram for the surface transition of the semi-infinite generalized BEG model should present the same topology. This enhances our expectation for the possibility of formation of a normal, $^4$He rich, film near the wall in some concentration (temperature range). Moreover, we did not prove that the long-range nature of the wall-mixture interactions is irrelevant for the details of the transitions. It is difficult to investigate this effect by means of the techniques we used, but we do not believe that it seriously affects the picture presented here.

More accurate experimental investigations of the whole phase diagram would be welcome. The determination of the order of the transition and the detailed analysis of the region where the nature of the transition changes are the most urgent questions. Answers to them will determine the direction of further theoretical studies.

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References


[12] After the completion of this work we received a preprint of D. McQueeney, G. Agnolet and J. D. Reppy on the measurements of the superfluid order near a wall in $^3$He-$^4$He mixtures with high $^3$He concentration. The authors have observed a continuous, Kosterlitz-Thouless type transition, with a sudden jump in the superfluid mass at the transition temperature. This two-dimensional phenomenon originates in the continuous symmetry of the superfluid order parameter and cannot be described by our simplified model.