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Dynamics of the phase variable in the Taylor vortex system

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Résumé. — Utilisant une méthode de développements en échelles multiples pour l'instabilité de Taylor-Couette, nous obtenons, dans un repère mobile particulier, une équation de diffusion pour la variable de phase du système de vortex. Le coefficient de diffusion transverse calculé est positif. L'équation de diffusion de la phase réduite au cas unidimensionnel semble adaptée pour l'interprétation des résultats expérimentaux de Snyder concernant la relaxation des états cellulaires dans l'écoulement de Taylor-Couette.

Abstract. — Using a multiple scale method for the Taylor-Couette instability, we obtain, in a suitable moving frame, a diffusion equation for the phase variable of the vortex system. The transversal diffusion coefficient $D_{\perp}$ is found to be positive. The phase diffusion equation restricted to the one-dimensional case seems to be relevant for interpreting Snyder's experiments on the relaxation of vortex systems in the Taylor-Couette flow.

1. Introduction.

It is well known that, concerning the first threshold, Taylor-Couette instability offers, in many respects, strong similarities with the Rayleigh-Benard instability : in both cases, the unstable modes are in the form of stationary, periodic flows involving virtually the same optimal wave-number. However, even in the idealization of infinite lengths, the similarities between the two flows cease to hold once the orientational properties of the cellular state are considered. In contrast with convective flows, the Taylor instability does not display any particular orientational invariance property : both the mean rotation and the shear of the mean flow tend to impose anisotropy. As a natural consequence, in Taylor experiments, the vortex system exhibits a preferred orientation close to the threshold.

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Another feature which differentiates the two flows is the presence of advection terms in the Taylor problem. Any disturbance with azimuthal dependence, will naturally tend to be advected in the azimuthal direction by the mean flow. This feature is not present in the Rayleigh-Benard problem.

These characteristics are involved in the problem of the stability of the cellular structures. In this paper we propose an analysis of the stability of the Taylor vortex system; the first step of the analysis will consist in deriving the dynamical equation for the slowly varying amplitude of the cellular motion. This derivation which parallels that of Newell and Whitehead for the Rayleigh-Benard problem, will be obtained by means of a classical multiple-scale perturbation technique. Then, by using the amplitude equation, we shall derive the governing equation for the phase variable of the system. This method restricts the study to a limited range of values of the Taylor number near threshold. However, one can expect that most of the features found herein will also be relevant for a more general class of perturbations.

2. Derivation of the amplitude equation.

The problem which we consider is the flow of a viscous incompressible fluid rotating between infinite concentric cylinders of radii R_1 and R_2 . The inner cylinder rotates with angular velocity Ω while the outer one is at rest. We further restrict ourselves to the small gap limit $d \ll R_1$ (where $d = R_2 - R_1$). In the steady state which corresponds to very low values of angular velocity Ω , the flow is laminar and the velocity is purely circumferential. In dimensionless form, the governing equations for the perturbed state read (to the lowest order in d/R_1) :

$$\frac{\partial u_x}{\partial t} + T^{1/2} V(x) \frac{\partial u_x}{\partial y} + (\mathbf{u} \cdot \nabla) u_x - \frac{1}{2} u_y^2 = -\frac{\partial p}{\partial x} + \Delta u_x + T^{1/2} V(x) u_y, \quad (1)$$

$$\frac{\partial u_y}{\partial t} + T^{1/2} V(x) \frac{\partial u_y}{\partial y} + (\mathbf{u} \cdot \nabla) u_y = \Delta u_y + T^{1/2} u_x, \quad (2)$$

$$\frac{\partial u_z}{\partial t} + T^{1/2} V(x) \frac{\partial u_z}{\partial y} + (\mathbf{u} \cdot \nabla) u_z = -\frac{\partial p}{\partial z} + \Delta u_z, \quad (3)$$

and

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \quad (4)$$

in which the dimensionless variables are $x = (R - R_1)/d$ (where R is the distance from the common axis of the cylinders), $y = (2 R_1/d)^{1/2} \theta$ (where θ is the azimuthal coordinate), $z = Z/d$ (where Z is the axial coordinate), and $t = \hat{T} \nu/d^2$ (where ν and $\hat{T}$ are the kinematic viscosity and the time respectively). In the above equations, the velocity reference for the x and z velocity components is ν/d while it is $(\nu/d)(R_1/2d)^{1/2}$ for the y velocity component; $V(x) = 1 - x$ is the dimensionless laminar velocity profile, $T = 2 \Omega_1^2 R_1 d^3/\nu^2$ is the Taylor number [1], $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the scalar Laplacian operator and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is the gradient operator.

The boundary conditions, associated to equations 1-4 are simply obtained by setting $u_x = u_y = u_z = 0$ on the walls $x = 0$ and $x = 1$. Imposing unrealistic free boundary conditions would not simplify in any respect the analysis of the problem in contrast with Rayleigh-Benard instability. For small Taylor numbers, the linear version of equations 1-4 yields an infinite denumerable set of eigenfunctions which exponentially decrease to zero as time increases, while as T passes through a critical value T_0 , the real part of an eigenvalue becomes positive. The corresponding unstable mode is axisymmetric ($\partial/\partial y = 0$), stationary and involves an optimal wave-number a_0 . This state corresponds to the well-known Taylor vortices.

Above the critical point, the cellular structure is allowed to exhibit slow variations in time and space provided that the discrepancy $T - T_0$ is small enough. Using the multiple scale approach, we introduce the following expansions :

$$\begin{aligned}\frac{\partial}{\partial t} &= \varepsilon^{1/2} \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2} + \dots, \\ \frac{\partial}{\partial y} &= \varepsilon^{1/2} \frac{\partial}{\partial y_1} + \varepsilon \frac{\partial}{\partial y_2} + \dots,\end{aligned}\quad (5)$$

and

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \varepsilon^{1/2} \frac{\partial}{\partial z_1} + \varepsilon \frac{\partial}{\partial z_2} + \dots,$$

in which

$$\varepsilon = (T^{1/2} - T_0^{1/2})/T_0^{1/2}.$$

The corresponding coherence lengths in both the azimuthal and axial directions are $O(\varepsilon^{-1/2})$, while the characteristic time scale near onset is $O(\varepsilon^{-1/2})$. This perturbation scheme is partially imposed by the presence of advection terms in equations 1-3. The new characteristic time scale which is introduced is much smaller than that related to the critical slowing down.

Accordingly the flow is expanded into series of the form

$$\mathbf{u} = \varepsilon^{1/2} \mathbf{u}^{(1)} + \varepsilon \mathbf{u}^{(2)} + \varepsilon^{3/2} \mathbf{u}^{(3)} + \dots, \quad (6)$$

each term being in turn developed into Fourier series in the form

$$\mathbf{u}^{(n)} = \sum_{l=-\infty}^{l=+\infty} \hat{\mathbf{u}}_l^{(n)} e^{il a_0 z_0}, \quad (7)$$

which recovers the basic periodicity of the structure at criticality.

Returning now to equations 1-4, and using expansions 5-7, we obtain an infinite hierarchy of linear equations which can be solved iteratively. To order $n = 1$, $l = 1$, we are facing again the instability problem which allows for the calculation of the basic Taylor vortex flow. This flow will further be denoted by velocity vector $\tilde{\mathbf{u}}_1^{(1)} = A(y_1, z_1, t_1, \dots) \mathbf{u}_1^{(1)}(x)$, where A is the slowly varying complex amplitude of the cellular flow, and $\mathbf{u}_1^{(1)}(x)$ is the normalized eigenfunction of the linearized problem. There is no difficulty at order $n = 2$, $l = 0$: we obtain a unique solution for the velocity field while the pressure is defined through an arbitrary function of y . This indeterminacy on the pressure is a simple consequence of the narrow gap assumption (see Eq. 2); the arbitrary function can be set equal to zero without loss of generality. The problem would be more complex in the wide gap case. However, even in this case, and in contrast with Rayleigh-Benard instability, the extra drift velocity that one can introduce at step $n = 2$, $l = 0$ is not a singular term, so that it is relevant only at the next order.

The first non trivial relation arises at order $n = 2$, $l = 1$. We obtain an equation of the form :

$$\frac{\partial A}{\partial t_1} + T_0^{1/2} v_0 \frac{\partial A}{\partial y_1} = 0, \quad (8)$$

in which we have

$$v_0 = \frac{\langle V(x) \tilde{\mathbf{u}}_1^{(1)} \cdot \mathbf{u}_1^{(1)} - T_0^{-1/2} \tilde{p}_1^{(1)} u_{y1}^{(1)} \rangle}{\langle \tilde{\mathbf{u}}_1^{(1)} \cdot \mathbf{u}_1^{(1)} \rangle},$$

where the bracket means integration throughout the gap width, and $\tilde{u}_1^{(1)}, \tilde{p}_1^{(1)}$ are the eigenfunctions of the adjoint operator of the linearized problem.

On physical grounds, v_0 corresponds to a propagation velocity; condition 8, merely expresses that any disturbance of the vortex system propagates along the mean flow at constant speed $T_0^{1/2} v_0$. Thus it is natural to introduce a new frame of reference which moves at velocity $T_0^{1/2} v_0$ by setting :

$$y'_1 = y_1 - T_0^{1/2} v_0 t_1$$

and

$$t'_1 = t_1$$

so that condition 10 becomes

$$\frac{\partial A}{\partial t'_1} = 0.$$

In this new frame of reference, the amplitude of the vortex system looks, to first order, stationary.

The solvability condition applied at order $n = 3, l = 1$ yields the amplitude equation in the form (in a reference frame moving at constant velocity $T_0^{1/2} v_0$) :

$$\frac{\partial A}{\partial t_2} = c_0 A + c_1 \frac{\partial^2 A}{\partial z_1^2} + ic_2 \frac{\partial^2 A}{\partial y_1' \partial z_1} + c_3 \frac{\partial^2 A}{\partial y_1'^2} - c_4 A |A|^2, \quad (9)$$

in which c_0, c_1, c_3 and c_4 are real coefficients. Equation 9 yields, to the lowest order, the governing equation for the complex amplitude of Taylor-vortex flows. In the rigid frame of reference which we consider, this equation involves both diffusion and propagation processes. The governing equation for the amplitude displays, as it should, the translational invariance of the problem. However, equation 9 includes anisotropic terms which account for the lack of orientational invariance of the Taylor problem. We shall study such terms in more detail.

The numerical calculation of the coefficients in equation 9 yields :

$$\begin{aligned} v_0 &= 0.5261, & c_0 &= 26.16 \\ c_1 &= 0.9837, & c_2 &= 0.3948 \\ c_3 &= 2.609 & \text{and } c_4 &= 40.2 \quad (\text{for normalization condition } v_1^{(1)}(\frac{1}{2}) = 1). \end{aligned}$$

The values for c_0, c_1 and c_4 are in excellent agreement with those calculated by previous authors [2, 3]. The value of v_0 corresponds to a propagation speed very close to the mean flow velocity.

3. Dynamics of the phase variable.

It is of interest to state the evolution equation for the phase variable of a Taylor-vortex system similarly as in the Rayleigh-Benard problem (see (4)). Generally speaking, let $\mathbf{u}(z)$ be the velocity field associated to a perfect system of Taylor cells between infinite cylinders. Due to translational invariance, if $\mathbf{u}(z)$ is a solution then $\mathbf{u}(z + \varphi)$ is also a solution when φ is constant. Seeking solutions in the form $\mathbf{u}(z + \varphi)$ where φ is now a slowly varying function of y, z and t leads to non trivial dynamics on the phase variable of the system. The corresponding dynamical condition is representative of the stability properties of the vortex system against phase perturbations. In this section, we shall develop such a calculation by using amplitude equation 9 rather than the full equations of the problem 1-4. This procedure restricts the range of validity of the analysis to the immediate vicinity of the threshold. However the calculations are considerably simpler and one can expect that this basic features found herein will also be relevant for a more general class of perturbations.

Returning to dimensionless coordinates x, y, z , and t , amplitude equation 9 becomes :

$$\frac{\partial A}{\partial t} = c_0 \varepsilon A + c_1 \frac{\partial^2 A}{\partial z^2} + ic_2 \frac{\partial^2 A}{\partial y' \partial z} + c_3 \frac{\partial^2 A}{\partial y'^2} - c_4 A |A|^2, \quad (10)$$

where $y' = y - T_0^{y_2} v_0 t$ and in which A has been replaced by $\varepsilon^{-1/2} A$. The strating point is a system of stationary axisymmetric vortices with wave-number $a_0 + \delta$, where δ is assumed to be small compared with a_0 .

δ is a measure of the compression (or dilatation) of the cellular structure which we consider. Thus we look for amplitude $A_0(z)$ under the form

$$A_0(z) = A_{00} e^{i\delta z}.$$

From equation 19, A_{00} is found to be :

$$A_{00} = \left(\frac{c_0 \varepsilon - c_1 \delta^2}{c_4} \right)^{1/2}$$

$A_0(z)$ is equal to zero when $c_0 \varepsilon = c_1 \delta^2$, which is precisely the local equation of the neutral curve in plane (δ, ε) . We further look for solutions under the form $A = A_0(z + \varphi(y', z, t))$ where φ is a slowly varying function of y', z and t ; φ , as well as its derivatives, is real.

Accordingly, we expand A into series of the form

$$A = A_0(z) + \frac{\partial A_0}{\partial z} \varphi + A_1(z) \frac{\partial \varphi}{\partial z} + A_2(z) \frac{\partial \varphi}{\partial y'} + A_3(z) \frac{\partial \varphi}{\partial t} + 0(\nabla \nabla \varphi) + \dots$$

in which $\nabla \varphi$ is the expansion parameter. At order zero in $\nabla \varphi$, we obtain :

$$\varphi A_0 \frac{\partial A_0}{\partial z} = 0, \quad (11)$$

where A_0 is the operator defined by :

$$A_0 = c_0 \varepsilon + c_1 \frac{\partial^2}{\partial z^2} - c_4 (A_0^2(*) + 2 |A_0|^2) \quad (12)$$

where $*$ denotes the complex conjugate. Equation 12 is clearly satisfied for any value of φ so that no condition is imposed on the vortex system, taken as a whole. This merely results from the translational invariance of the problem. At first order in $\nabla \varphi$, one gets :

$$A_0 \left(A_1 \frac{\partial \varphi}{\partial z} + A_2 \frac{\partial \varphi}{\partial y'} + A_3 \frac{\partial \varphi}{\partial t} \right) = - \frac{\partial^2 A_0}{\partial z^2} \left(2 c_1 \frac{\partial \varphi}{\partial z} + ic_2 \frac{\partial \varphi}{\partial y'} \right) + \frac{\partial A_0}{\partial z} \frac{\partial \varphi}{\partial t}. \quad (13)$$

Applying the solvability condition leaves a relation of the form :

$$\frac{\partial \varphi}{\partial t} + v_1 \frac{\partial \varphi}{\partial y'} = 0, \quad (14)$$

where

$$v_1 = c_2 \delta. \quad (15)$$

On physical grounds, (14) means that any transverse phase disturbance propagates in the direction of the mean flow with velocity v_1 .

In contrast with the velocity v_0 , v_1 is closely related to the compression (or dilatation) of the cellular structure. Similarly as in section 2, it is natural to introduce a new coordinate y'' which allows for the description of the flow in a rigid frame of reference, now moving at velocity $T_0^{1/2} v_0 + v_1$. In this new frame of reference, the phase disturbances look, to first order, stationary.

Turning to the terms of order $\nabla\nabla\varphi$, and applying the solvability condition, yields a relation of the form :

$$\frac{\partial\varphi}{\partial t} = D_{\parallel} \frac{\partial^2\varphi}{\partial z^2} + D_{\perp} \frac{\partial^2\varphi}{\partial y''^2}, \quad (16)$$

where

$$D_{\parallel} = c_1 \frac{\delta_{\parallel}^2 - 3\delta^2}{\delta_{\parallel}^2 - \delta^2}$$

and

$$D_{\perp} = c_3,$$

in which we have

$$\delta_{\parallel}^2 = \frac{c_0}{c_1} \varepsilon.$$

Equation 16 yields a diffusion equation for the dynamics of the phase variable. The stability of the flow is ensured when all the diffusion coefficients are positive, while a negative coefficient means instability. Since we have $c_1 > 0$, $D_{\parallel}$ is negative in the wave-number range defined by

$$|3^{-1/2} \delta_{\parallel}| < |\delta| < |\delta_{\parallel}|$$

and is positive elsewhere. This corresponds to the Ekhaus instability [5]. Since c_3 is positive, the stability of the vortex system against transversal disturbances is ensured. This contrasts with the instability of convective flows against slight tilting of the roll system (zig-zag instability).

4. Some comparison with Snyder's experiment.

Snyder [6] performed very interesting measurements on the wavelength of Taylor-vortices in an apparatus where the gap width is equal to the inner radius. He found that the relaxation time of the vortex system is approximately $0.15 L^2/\nu$ where L is the length of the vortex column. The experimental procedure consisted first in attaining a steady supercritical state (defined by the Taylor number and the cell wavelength), after which a sudden increase in the angular velocity of the cylinder is applied. Snyder observed that the cell wavelength changes with time and then stabilizes at a definite level. The dependence of the wavelength variations with time is shown (in a normalized form) in figure 1, for different initial and final values of the Taylor number.

It is possible to interpret such experiments by using phase diffusion equation 16, restricted to the one-dimensional case i.e. with $\partial/\partial y'' = 0$. In term of the original coordinates Z and $\hat{T}$, equation 16 reads :

$$\frac{\partial\varphi}{\partial\hat{T}} = D_{\parallel} \nu \frac{\partial^2\varphi}{\partial Z^2}, \quad (17)$$

which can be integrated provided that some boundary conditions are given. Since the initial and final steady states involve two distinct wavelengths which are close to their optimal value, one can impose the following conditions on φ at times $\hat{T} = 0$ and $\hat{T} = \infty$ respectively :

$$\varphi = 0 \quad \text{for} \quad \hat{T} = 0$$

Fig. 1. — Adjustment of the Taylor-vortex system as a function of dimensionless diffusion time, in normalized form : comparison between the Snyder experiment and the phase diffusion model. The corresponding values of discrepancy ε are : $\times$ 0.072; $\circ$ 0.41; $\square$ 1.03; $\bullet$ 1.32; $+$ 1.03; the full line represents the theoretical results, calculated from equation 19, with $c_1 = 1.15$ and $Z \simeq 0$.

and

$$\varphi = 2\pi \frac{\Delta\lambda_\infty}{\lambda_{\text{opt}}^2} Z \quad \text{for } \hat{T} \rightarrow \infty, \quad (18)$$

where $\Delta\lambda_\infty$ is the final wavelength variation of the vortex system and λ_{opt} is the optimal wavelength. The final condition in 18 is valid only when the wave-number is close to its optimal value during the relaxation process. This is precisely the case in Snyder's experiments. A relation between $\varphi(Z, \hat{T})$ and wavelength variation $\Delta\lambda(Z, \hat{T})$ holds in regions where φ varies slowly with Z . We find :

$$\Delta\lambda \simeq \frac{\lambda_{\text{opt}}^2}{2\pi} \frac{\partial\varphi}{\partial Z}.$$

Solving equation 17 subjected to conditions 18 yields the following expression for $\Delta\lambda$:

$$\Delta\lambda = \Delta\lambda_\infty \left(1 - e^{-\tau} \cos \frac{2\pi Z}{L} + e^{-4\tau} \cos \frac{4\pi Z}{L} - \dots \right) \quad (19)$$

where

$$\tau = 4\pi^2 D_{\parallel} \nu \hat{T} / L^2.$$

Far from the end plates ($Z \sim 0$), curves $\Delta\lambda(Z, \hat{T})/\Delta\lambda_\infty$ reproduce fairly well those obtained by Snyder : figure 1 shows that a reasonable fitting is obtained with $c_1 = 1.15$. This estimate is consistent with that which can be deduced from the experimental curves of Pfister *et al.* [7], and the value calculated by Yahata [8]. However it is difficult to draw any definite conclusion about the degree of accuracy of the phase diffusion model.

The diffusion coefficient $D_{\parallel}$ generally depends on the initial and final conditions imposed on the vortex system. It turns out that, owing to the experimental data of Snyder, $D_{\parallel}$ is left virtually unaffected by changes in the initial and final conditions. This explains why the theoretical values lie virtually on a single curve ; in any event, the statement of Snyder that the adjustment time is independent of the initial and final steady states should be probably revised.

5. Conclusion.

The present results, concerning the phase variable dynamics of the Taylor-vortex system, seem to be relevant for a more or less quantitative interpretation of Snyder's observations. Up to now there is no experimental data directly connected with transversal dynamics of the

phase variable. The fact that, even between very long cylinders, Taylor-vortex systems exhibit spatial order may have some link with the result about the stability of the cellular structure against transversal disturbances. However, this point clearly requires additional information [9].

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