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Directed percolation in two dimensions : renormalization group approach (*)

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Résumé. — Nous présentons un calcul de groupe de renormalisation de la percolation dirigée sur un réseau carré anisotrope, sur lequel les liens horizontaux (resp. verticaux) sont présents avec une concentration x (resp. y). Nous étudions l'existence d'un chemin infini partant de l'origine et dirigé dans une direction d'angle θ donné dans le diagramme de phase xy . Notre méthode reproduit les résultats exacts pour $x = 1$ (ou $y = 1$) pour tout θ . Nous effectuons aussi un calcul explicite pour $\theta = 45^\circ$ et obtenons la frontière critique et le comportement de *cross-over* de l'exposant de la longueur de corrélation ν le long de la frontière.

Abstract. — We present a renormalization group calculation for the directed percolation problem in an anisotropic square lattice, in which the horizontal (vertical) bonds are present with a concentration x (concentration y). The existence of an infinite path from the origin along a given direction θ is investigated in a xy phase diagram. Our method reproduces the exact known results for the $x = 1$ (or $y = 1$) limit for any θ . We also present an explicit calculation for the $\theta = 45^\circ$ case, and obtain the critical frontier and the expected cross-over of the correlation length exponent ν along this frontier.

The problem of directed percolation has a broad range of applications, from Reggeon field theory to chemistry and biology [1]. Particularly in two dimensions, this problem has been theoretically studied by Monte Carlo [2], series [3] and phenomenological renormalization techniques [4]. There are some good estimates for the critical concentrations and correlation length exponents, mostly from calculations performed on the square lattice.

We briefly recall the main features of the isotropic directed percolation problem. Let us consider a square lattice with oriented bonds, the horizontal ones to the right, and the vertical ones up. These bonds can be present (absent) with probability $p(1 - p)$. Figure 1 represents schematically, for three distinct values of p , the island of sites which can be reached from the origin, passing only through existent bonds and obeying their orientations. For $p = p_1 < p_c$, such an island is finite. For $p = p_c$, the island is infinite if we measure its length ξ_{45} along the diagonal direction ($\theta = 45^\circ$), but it is finite in all other directions. Near p_c , this correlation length diverges as $\xi_{45} \sim (p_c - p)^{-\nu}$. For $p = p_2 > p_c$, the island is infinite along a range of directions θ around 45° . The correlation

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Fig. 1. — Schematic representation of the island of sites connected to the origin (sites which can be reached from the origin). The lines with double traces are asymptotically parallel.

length ξ_θ along $\theta \neq 45^\circ$ diverges as $\xi_\theta \sim (p_\theta - p)^{-\nu_1}$, at a critical concentration $p_\theta > p_c$ depending on θ . Intuitively, it is reasonable to expect that the critical exponent ν_1 be different from ν , for ν corresponds to the process of the island size becoming infinite (along $\theta = 45^\circ$), whereas ν_1 corresponds to the angular broadening of the already infinite island. Also, we expect that the value of ν_1 be the same for all directions $\theta \neq 45^\circ$. The values of p_c , p_θ , ν and ν_1 have been estimated [1, 4] by using the phenomenological renormalization group approach, and the results are in complete agreement with the previous intuitive analysis.

Domany and Kinzel [1] formulated an alternative approach to the problem : instead of varying the angle θ , they proposed to take a fixed direction (for instance $\theta = 45^\circ$), allowing the horizontal and vertical bond concentrations (x and y respectively) to be different. Varying the anisotropy parameter y/x , the islands of figure 1 will rotate, and this rotation is supposed to yield the same effect as varying the angle θ in the isotropic problem. Domany and Kinzel [1] were able to calculate exactly the probability of connection $P_{mn}(x, y)$ between the origin and the lattice site located n units right and m units up (point P in Fig. 2), for the limits $x = 1$ or $y = 1$. The critical exponent $\nu_1 = 2$ can be extracted exactly from these results, and this value is thought to be valid also for $x \neq 1$ and $y \neq 1$. The critical concentration x_c for $y = 1$ (or y_c for $x = 1$) is also exactly known as a function of θ (see Eq. 4 below).

Fig. 2. — Cell used for calculating $f_m^n(x, y)$ (see text). The direction θ corresponding to this cell is that of the line OP.

For the anisotropic problem at a fixed direction θ , we can imagine a phase diagram in the xy plane with a critical frontier, like that of figure 3 (or 4). The purpose of the present work is to present a method for constructing such a diagram, and for calculating the critical exponents ν and ν_1 . In order to do so, we define a class of functions $f_m^n(x, y)$ representing the probability of vertical spanning of a rectangular $m \times n$ cell, starting from the origin (see Fig. 2). For the computation of $f_m^n(x, y)$ we must sum the probabilities of all cell configurations for which there is a path linking the origin 0 to any point of row $m + 1$ (outside the cell). The probability of each configuration is $x^i(1-x)^{M-i}y^j(1-y)^{N-j}$, where $i(j)$ is the number of horizontal (vertical) bonds present in that particular configuration, and $M(N)$ is the total number of horizontal (vertical) bonds in the cell. Note that the probability of horizontal spanning of the same $m \times n$ cell is $f_n^m(y, x)$.

We now follow a position space renormalization group approach, by renormalizing a rectangular $mb \times nb$ cell into a smaller $m \times n$ cell, through a scale factor b . In doing so, we are simulating the behaviour of the system along a fixed direction θ such that $\tan \theta = m/n$. The concentrations x and y are respectively renormalized into x' and y' , according to equation 1.

$$\left. \begin{aligned} f_m^n(x', y') &= f_{mb}^{nb}(x, y) \\ f_n^m(y', x') &= f_{nb}^{mb}(y, x) \end{aligned} \right\} \quad (1)$$

Fig. 3. — Flow diagram for $\theta = 45^\circ$. The arrows indicate the local flow direction. Stable fixed points are represented by squares, semistable ones by full circles, and the totally unstable fixed point by an open circle.

Fig. 4. — Flow diagram for $\theta = \tan^{-1} 2 = 63^\circ$.

We are able to compute exactly the functions $f_m^n(1, y)$ for any pair (m, n) , by a procedure similar to that used by Domany and Kinzel [1]. We quote the result in equation 2

$$f_m^n(1, y) = 1 - (1 - y)^n \sum_{k=1}^m \frac{(n + k - 2)!}{(k - 1)!(n - 1)!} y^{k-1}. \quad (2)$$

Comparing with Domany and Kinzel's result, we note that $f_m^n(1, y) = 1 - P_{mn}(1 - y)$. In particular, taking the limit $n = \alpha m \rightarrow \infty$, where $\alpha = \text{ctg } \theta$ is a constant, we have :

$$\lim_{m \rightarrow \infty} f_m^{\alpha m}(1, y) = \begin{cases} 0 & \text{for } y < y_c \\ 1/2 & \text{for } y = y_c \\ 1 & \text{for } y > y_c, \end{cases} \quad (3)$$

$$y_c = (1 + \alpha)^{-1}. \quad (4)$$

Equation 4 coincides with the exact result [1] for the end points of the critical frontier. Thus, our method gives the exact location for these points, at least in the large-cell limit. The exact value for the exponent $\nu_1 = 2$ is also achieved at these points, in the same limit.

We have not been able to compute the function $f_m^n(x, y)$ for arbitrary values of x, y, m and n , but it is possible for instance to take $\theta = 45^\circ$ and renormalize a 2×2 cell into a 1×1 cell through a $b = 2$ scale factor. This simple calculation, however, allows us to reach conclusions on the cross-over between isotropic and anisotropic situations. Equations 1 become equations 5.

$$\left. \begin{aligned} x' &= x^2 + y(2x^2 - x^3 - x^4) + y^2(-x^3 + x^4) \\ y' &= y^2 + x(2y^2 - y^3 - y^4) + x^2(-y^3 + y^4) \end{aligned} \right\} \quad (5)$$

The flow diagram of equations 5 is shown in figure 3. There are two stable fixed points, one at the origin corresponding to the disordered phase (inexistence of infinite paths), and the other at $x = y = 1$ corresponding to the completely ordered phase. These two phases are limited by a critical frontier BC which separates the points (x, y) that flow to the origin by successive renormalizations from those points (x, y) that flow to $x = y = 1$. Points B and C are at the correct coordinates $(1, 1/2)$ and $(1/2, 1)$, according to the exact relation 4. The exact position of the isotropic point A is not known, but it is estimated to be around $x = y = 0.64$ by the various methods quoted earlier. Point A of figure 3 is located at $x = y = 0.67$, which can be considered an excellent result for such a small 2×2 cell. Moreover, the flow along the critical line BC shows that the critical behaviour at the totally unstable fixed point A differs from the one at *all* other critical points on the line, because these others are attracted by the semistable fixed points B or C. This feature is in agreement with the hypothesis [1] that $\nu_1 = 2$ is the critical exponent along all the critical line, except at A where $\nu = 1.73$ [1, 4].

The numerical results obtained at the fixed point A for 2×2 , 3×3 and 4×4 cells (renormalized into the 1×1 one) are displayed in table I. Note that the values for ν are improved for increasing cells, but the locations of A are not. This is a common feature of small-sized-cell calculations [5]. The eigenvalue λ_2 along the critical line direction seems to approach unity, indicating a possible marginality (note that λ_2 does not increase with the cell size). Corroborating to this marginality hypothesis, one can show that the weak eigenvalue $\lambda_3 < 1$ at the other two fixed points B or C tends *exactly* to unity : by counting all configurations with no more than one missing vertical bond, in a $m \times m$ cell, we obtain $\lambda_3 = 1 - 1/2^m$ that tends to unity *from below* in the limit of large cells. An interesting feature of the marginality of λ_3 is that it will tend to unity *from above* if one renormalizes a $m \times m$ into a $n \times n$ ($m > n > 1$) cell : in this case we would have $\lambda_3 = (1 - 1/2^m)/(1 - 1/2^n)$; and the flow along the critical line BC in figure 3 would be inverted.

Table I. — *Position of the isotropic fixed point A, and the critical exponent ν corresponding to the renormalization group calculations.*

Cell	$x = y$	λ_1	$\nu = \ln b / \ln \lambda_1$	λ_2
2×2	0.671	1.624	1.430	1.099
3×3	0.680	2.057	1.523	1.103
4×4	0.684	2.390	1.591	1.089
accepted values [1-4]	0.64		1.73	

Using our *finite-cell* renormalization, we are approaching a *logarithmic* flow (if the marginality hypothesis is correct) along the critical line by successive *power-law* flows corresponding to exponents converging to unity in the limit of large cells. Because the *flow direction* is *conserved* in those processes (one for $m \times m$ to $n \times n$ renormalizations, or the opposite direction for $m \times m$ to 1×1 renormalizations), we must choose the process that gives the correct flow direction from the beginning. In this manner, we obtain the correct cross-over behaviour at point A (and not at B or C) [6].

Most renormalization group calculations do not take into account *two independent* scaling fields in a 2-dimensional space of parameters, and consequently do not describes correctly the cross-over at point A. Domany and Kinzel [1], for example, use the phenomenological approach by renormalizing strips taken along different directions, for the isotropic problem. They have obtained $\nu = 1.732$ at $\theta = 45^\circ$, and values close to $\nu = 2$ at all other tried directions. In the anisotropic problem, this would correspond to restrict the flow of figure 3 to a straight line passing through the origin (y/x constant). This kind of truncation of the space of parameters usually introduces spurious fixed points and bad values for the exponents [7]. In our case, *all* the critical line of figure 3 would be formed by fixed points in such a parametric renormalization approach. Fortunately, the eigenvalue λ_2 along this line seems to be close to unity (see Table I), indicating that the critical points are *quasi* fixed (possibly they flow logarithmically). This fact enables Domany and Kinzel to predict the value $\nu_1 = 2$ along all the critical line (except A), based on their good numerical results. Ikeda [7] did not have the same chance for the undirected percolation problem.

There are two other semistable fixed points, one at $(x, y) = (1, 0)$ and the other at $(x, y) = (0, 1)$ (see Fig. 3). Both give the exactly known critical concentration $p_c = 1$ for one-dimensional percolation (directed or not) problem. Further, the correlation length exponent extracted from these two fixed points is $\nu_0 = 1$, which coincides with the exact known one-dimensional value, and so, we conclude that our method gives also the correct cross-over between two and one-dimensional behaviour.

The isomorphism proposed by Domany and Kinzel between the isotropic problem with varying directions θ and the anisotropic problem with a fixed arbitrary direction θ is very plausible, and can be tested by applying our method for other values of θ ($\neq 45^\circ$). We have been able to renormalize a 4×2 cell into a 2×1 smaller one, through a scale factor $b = 2$. Equations 1 become equations 6.

$$\begin{aligned}
 x' + y'(x' - x'^2) = & x^2 + y(2x^2 - x^3 - x^4) + \\
 & + y^2(3x^2 - 4x^3 - x^4 + x^5 + x^6) + \\
 & + y^3(4x^2 - 8x^3 + 5x^5 + x^6 - x^7 - x^8) \\
 & + y^4(-4x^3 + 6x^4 + 3x^5 - 5x^6 - 3x^7 + 3x^8) \\
 & + y^5(-2x^3 + 6x^4 - 3x^5 - 7x^6 + 9x^7 - 3x^8) \\
 & + y^6(-x^3 + 5x^4 - 10x^5 + 10x^6 - 5x^7 + x^8) \\
 y'^2 = & y^4 + x(4y^4 - y^5 - y^6 - y^7 - y^8) + x^2(-3y^5 - y^6 + y^7 + 3y^8) + \\
 & + x^3(2y^6 + y^7 - 3y^8) + x^4(-y^7 + y^8). \quad (6)
 \end{aligned}$$

The flow diagram of equations 6 is shown in figure 4. Points B and C are located at $y_c = (1 + \sqrt{17})/8 = 0.64$ and $x_c = (7 - \sqrt{17})/8 = 0.36$ respectively. The correct values given by equation 4 are $y_c = 2/3 = 0.67$ and $x_c = 1/3 = 0.33$. The correlation length exponent ν_1 has the same value $\nu_1 = 1.78$ at B and C (we recall that, for large cells, we obtain the correct value $\nu_1 = 2$). The correct cross-over between two and one-dimensional behaviour is also achieved. Surprisingly enough, the internal totally unstable fixed point A is not present. However, we have reasons to believe that this point A will appear if we use large cells : in figure 5 we have plotted the displacement of the critical points along the frontier under one step renormalization, for equations 5 (Fig. 5a) and equations 6 (Fig. 5b). The minimum (in absolute value) observed at the middle of figure 5b is not a common feature in renormalization group calculations. We believe that, for large cells, the curve will cross the horizontal axis, giving origin to two new fixed points (call them A at the left, and B' at the right). If B' will tend to B for large cells, then figure 5b will have the same form of figure 5a. Unfortunately, we cannot prove this point, and a confirmation of the isomorphism speculated by Domany and Kinzel [1] can be made only by using large cells, within the present framework.

In summary, we have constructed a position space renormalization group for the directed percolation problem in a square anisotropic lattice. The exact known results [1] for this problem are the correlation length critical exponent ν_1 and the critical vertical bond concentration y_c as a function of the observation direction θ , in the particular case of full presence ($x = 1$) of the horizontal bonds. Our renormalization group approach reproduces all these exact results. Furthermore, applying our method to the specific case of $\theta = 45^\circ$, we obtained a critical frontier in the whole xy space and the flow direction confirms the hypothesis [1] of a cross-over at the isotropic critical point. Also, the position obtained for this point and its correlation length exponent are in good agreement with previous estimated values. Also, our calculations indicate a possible marginal behaviour along the critical line. We think that this possibility must be investigated further, specially in the context of Reggeon field theory [8] where marginality is a common feature.

Fig. 5. — Plot of the displacement of critical points along the critical frontier for a) $\theta = 45^\circ$ and b) $\theta = \text{tg}^{-1} 2 = 63^\circ$.

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