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Random walk on a two-dimensional random environment

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Résumé. — Nous considérons un modèle de marche aléatoire sur un réseau bidimensionnel et sur des rubans où les probabilités de saut d'un site à un de ses premiers voisins sont aléatoires. Dans le cas d'une distribution de probabilité non symétrique, à savoir d'une marche avec direction privilégiée, la vitesse du déplacement ne peut s'annuler que pour des distributions de probabilité très particulières.

Abstract. — We analyse a model of random walk on a two-dimensional lattice and on a strip where the probabilities of hopping from a site to one of its nearest neighbours are randomly given. In the case of a non-symmetric probability distribution, when the walk has a preferred direction, we find that the drift velocity may vanish only for very particular probability distributions, contrary to the one-dimensional case.

1. Introduction.

The problem of the random walk on disordered lattices has been widely studied in one dimension both in the symmetric case [1, 2] and in the non-symmetric case [3], by analytical and numerical methods. Nevertheless, in two-dimensions only the symmetric case has so far been considered [4].

A random chain is defined by the probability distribution $\rho(p_i)$ to hop right p_i and to hop left $q_i = 1 - p_i$ from the i site of the chain.

A remarkable result was obtained by Sinai [2] who showed that the averaged displacement $\langle \bar{R}^2(t) \rangle^{1/2} \sim \ln^2 t$ for distributions $\rho(p_i)$ satisfying the condition $\langle \ln(p_i/q_i) \rangle = 0$. In the following we indicate the average on the distribution ρ by $\langle . \rangle$ and the time average on a single realization by $\overline{.}$; here t is the time or the number of steps in a discrete version of the model.

On the other hand Derrida and Pomeau [3] studied the non-symmetric case $\langle p \rangle > \langle q \rangle$ and they found a drift on the right only when the stronger condition $(\langle 1/p \rangle)^{-1} > 1/2$ is satisfied. In the intermediate region between $\langle p \rangle > 1/2$ and $(\langle 1/p \rangle)^{-1} < 1/2$ it is possible to show that $\langle \bar{R}(t) \rangle \sim t^x$ with $x < 1$ and a vanishing drift velocity $V = \lim_{t \rightarrow \infty} \langle \bar{R}(t) \rangle / t$. Moreover x vanishes when p approaches $1/2$ in agreement with the Sinai prediction.

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It is our purpose to show that in a two-dimensional, non-symmetric random lattice the velocity may vanish only imposing very strong conditions on the probability distribution.

2. The model.

We consider a natural two-dimensional generalization of the random chain : to each site (i, j) of a square lattice the probabilities $\pi_{ij}^{(\mu)}$ of hopping to one of the four nearest neighbours μ are given.

Let us call $q_{ij}(p_{ij})$ the probability of a left (right) hop, $d_{ij}(u_{ij})$ the probability of a down (up) hop and $q + p + d + u = 1$. A non-symmetric distribution $\rho(\pi_{ij})$ has e.g. to satisfy :

$$\begin{aligned} \langle p_{ij} - q_{ij} \rangle &> 0, \\ \langle u_{ij} - d_{ij} \rangle &\geq 0. \end{aligned} \quad (1)$$

We are interested in finding which further conditions one has to impose on $\rho(\pi)$ so that the drift velocity towards the up-right region of the lattice does not vanish. We shall confine ourselves to a particular model in which few of the sites, with probability $P_B \ll 1$, are very uneasy to climb over. These sites, having a hop probability $\pi_{ij}^{(\mu)} = \bar{p} \ll 1$ towards one direction μ , can originate « barriers » (i.e. open, reflecting lines) as well as « traps » (i.e. close lines enclosing a region from which it is difficult to escape).

On the contrary, all the other sites have probabilities $\pi_{ij}^{(\mu)}$ close to the average value $\langle \pi^{(\mu)} \rangle$. In practice the motion of a particle in such a random lattice is described by a « nearly deterministic » walk hindered by the barriers and the traps. Now, in a « nearly deterministic » walk the « naive » velocity would be

$$V_x^{(n)} = \langle p - q \rangle ; \quad V_y^{(n)} = \langle u - d \rangle, \quad (2)$$

while the displacement in the x -direction after t steps will be, considering both traps and barriers :

$$\langle \bar{\Delta x}(t) \rangle = V_x^{(n)} \cdot t \cdot P_0 + \sum_{\{\gamma\}} V_x^{(n)} [t - \bar{t}(\gamma)] P(\gamma), \quad (3)$$

where P_0 is the probability to have neither barriers nor traps, while $P(\gamma)$ is the probability to have a given configuration of barriers and traps and $\sum_{\{\gamma\}} P(\gamma) = 1 - P_0$; $\sum_{\{\gamma\}}$ is the sum over all possible configurations and $\bar{t}(\gamma)$ is the slowing down time due to a given configuration γ averaged over a single realization. We rewrite (3) as :

$$\langle \bar{\Delta x}(t) \rangle = V_x^{(n)} \left[t - \sum_{\{\gamma\}} \bar{t}(\gamma) P(\gamma) \right]. \quad (4)$$

To estimate the sum over configuration in (4) we shall divide it into a sum on the barriers $\sum_{\{\gamma\}}^{(1)}$ and a sum on the traps $\sum_{\{\gamma\}}^{(2)}$, so that $\sum_{\{\gamma\}} \leq \sum_{\{\gamma\}}^{(1)} + \sum_{\{\gamma\}}^{(2)}$.

Let us consider the barrier case. We shall compute the sum on the configuration $\{\gamma\}$ by a sum on the length of the barriers $|\gamma|$. In fact, it is easy to see :

$$\sum_{\{\gamma\}}^{(1)} \bar{t}(\gamma) P(\gamma) \leq \sum_{\gamma \geq 1} \bar{t}_1(|\gamma|) n_1(|\gamma|), \quad (5)$$

where $n_1(|\gamma|)$ is the average number of barriers of length $|\gamma|$ met in a path of t steps and $\bar{t}_1(|\gamma|)$ is the slowing down time due to one barrier of length $|\gamma|$. As P_B is the probability that a « wall » site occurs, it follows that the probability of one $|\gamma|$ -barrier is $P_1(|\gamma|) \lesssim P_B^{|\gamma|}$ and so :

$$n_1(|\gamma|) \lesssim t P_B^{|\gamma|}. \quad (6)$$

On the other hand, we have called $\bar{P}$ the probability of a hop from a « wall » site to one of its nearest neighbours.

Then $t_1(|\gamma|) \sim \bar{P}^{-1}$ when $|\gamma|$ is large enough so that the barrier cannot be crossed by the diffusion process of the random walk.

In the one-dimensional case an exact estimate of the diffusion in a random walk with a constant drift, reflecting boundary and $p_i \equiv p$ for each i should give (6)

$$\bar{t}_1(|\gamma|) \sim (p - q)^{-2} \exp \{ \gamma \ln(p/q) \}.$$

Therefore we can assume :

$$\bar{t}_1(|\gamma|) \lesssim \text{Min} \{ \bar{P}^{-1}; e^{\ln(a) \cdot |\gamma|} V^{(n)-2} \} \quad (7)$$

where

$$a \sim \langle p \rangle / \langle q \rangle, \quad V^{(n)-2} = \text{Min} \{ V_x^{(n)-2}, V_y^{(n)-2} \}$$

and

$$\bar{t}_1(|\gamma|) \sim \bar{P}^{-1} \quad \text{for } |\gamma| > \gamma^* = \ln(\bar{P}^{-1}/V^{(n)-2})/\ln a.$$

Then we obtain by inserting (7) and (6) in (5) :

$$\sum_{\{\gamma\}}^{(1)} \bar{t}(|\gamma|) P(\gamma) \lesssim t V^{(n)-2} \int_1^{\gamma^*} e^{\gamma \ln(P_B a)} d\gamma + t \bar{P}^{-1} \int_{\gamma^*}^{\infty} e^{\gamma \ln P_B} d\gamma. \quad (8a)$$

Here $P_B \ll 1$ and $a > 1$ from (1).

Actually we are not interested in nearly deterministic walks with $\langle q \rangle \sim 0$, so that it is reasonable to suppose $a < P_B^{-1}$. It follows that :

$$\begin{aligned} \sum_{\{\gamma\}} \bar{t}(\gamma) P(\gamma) &\lesssim t V^{(n)-2} \int_1^{\infty} e^{-|\ln P_B a| \cdot |\gamma|} d|\gamma| \\ &\lesssim t V^{(n)-2} a \frac{P_B}{|\ln P_B a|} \sim 0(P_B) \cdot t. \end{aligned} \quad (8b)$$

Besides we have to require $V^{(n)-2} < P_B^{-1}$ in (8b), which implies $(a - 1)^2 > P_B$. In this limit we are not able to exclude that the slowing time due to the barriers becomes considerable and that the drift velocity vanishes. However such a transition to a zero speed is pushed to the symmetric case $a \sim 1$ as P_B has a small value.

We have now to estimate the slowing down time due to the traps. We shall use the notation of relation (5) where the subscript 2 instead of 1 indicates that we are dealing with « traps » instead of « barriers ». The simplest traps which are the most probable to occur are shown in figure 1 with their weights $P_2(|\gamma|)$.

Fig. 1. — The most probable traps with their weights (the coefficients c_i are numerical factors). The dotted lines indicate the « forbidden » directions ; the full lines indicate the « easy » directions.

As well as before $n_2(|\gamma|)$ is the average number of traps of perimeter $|\gamma|$ met in a walk of t steps, while the average time to escape from a trap is now of the order of the inverse of the « hop » probability, i.e. $\bar{t}_2(|\gamma|) \sim \bar{P}^{-1}$.

The same arguments which led to (5) and (6) allow to give a heuristic estimate of the sum on the traps :

$$\begin{aligned} \sum_{\{\gamma\}} \bar{t}(\gamma) P(\gamma) &\sim \sum_{|\gamma| \geq 2} \bar{t}_2(|\gamma|) n_2(|\gamma|) \\ &\sim t \bar{P}^{-1} \sum_{|\gamma| \geq 2} P_2(|\gamma|) \\ &\sim \text{const. } t \bar{P}^{-1} P_B^6. \end{aligned} \quad (9)$$

The last step in (9) follows from the obvious consideration that P_B^6 is the leading power in the sum, see figure 1.

Therefore the displacement in the x -direction will be :

$$\langle \bar{\Delta x}(t) \rangle \gtrsim t V_x^{(n)} \{ 1 - 0(P_B^6 \bar{P}^{-1}) - 0(P_B) \} \quad (10)$$

and the drift velocity of a « random-random » walk cannot vanish in 2 dimensions, almost for this particular barriers-and-traps model, if the condition

$$P_B^6 \bar{P}^{-1} \ll 1, \quad (11)$$

is satisfied, together with the constraint $1 + P_B^{1/2} < a < P_B^{-1}$.

Nevertheless we think that estimate (11) is too crude because we did not compute the numerical coefficients c_i for the probabilities $P_2(|\gamma|)$ in figure 1, which are very small. As $\langle 1/p^\alpha \rangle \approx P_B \bar{P}^{-\alpha}$, condition (11) holds when :

$$\left(\left\langle \frac{1}{p^\alpha} \right\rangle \right)^{-1} > \bar{P}^{\alpha-1/6}, \quad (12)$$

where the right hand side of the inequality is really small for $\alpha > 1/6$.

Even if the class of the distribution $\rho(\pi)$ giving our simple model may seem not too general, nevertheless we think that all the distributions reduce themselves to a « barriers-and-traps » $\rho(\pi)$ by coarse graining procedures (i.e. by considering larger lattices) and centre limit theorem arguments. However it seems to us that such a model, among all possible models of random walks, describes the easiest situation in which the drift velocity may vanish in spite of constraint (1).

We would also like to remark that the generalization to higher dimensions d is straightforward : condition (11) is replaced by $P_B^{4d-2} \bar{P}^{-1} \ll 1$ which becomes weaker and weaker when increasing d . In $d = 1$ one has to require $P_B \bar{P}^{-1} \approx \langle 1/p \rangle < 1$ and our estimate is in good agreement with the exact result of reference [3] $\langle 1/p \rangle < 2$.

3. Finite size effects.

We shall consider now a strip of infinite length in the x -direction and of width L in the y -direction. We may repeat our analysis to arrive at a similar conclusion as in (9), but we have to change the estimate (7) of $\bar{t}_1(|\gamma|)$. In fact it is not possible to turn around a barrier which cuts the strip in two parts. Then

$$\bar{t}_1(|\gamma|) \lesssim \begin{cases} \bar{P}^{-1} & |\gamma| \geq L, \\ V^{(n)-2} e^{|\gamma| \ln a} & |\gamma| < L. \end{cases} \quad (13)$$

And we obtain in the same manner the drift velocity

$$V \geq V^{(n)} \left\{ 1 - 0\left(\frac{P_B^L}{\bar{P}}\right) - 0\left(\frac{P_B^6}{\bar{P}}\right) - 0(P_B) \right\}. \quad (14)$$

This estimate follows from the obvious consideration that the effective barriers (cutting the strip) have probability P_B^L and the slowing down time will be of order $(\bar{P}^{-1} P_B^L) t$.

Finally we have to repeat that the $0(\bar{P}^{-1} P_B^6)$ term in (10), due to the traps, certainly gives a too high estimate of the correction to the « naive » velocity, while the $0(\bar{P}^{-1} P_B^L)$ term, due to the barriers which cut the strip, is rather accurate and can dominate for L which are small but however larger than 6.

4. An example.

We choose a particular distribution ρ to emphasize our results :

$$\begin{aligned} \rho(p_i) &= (1 - \varepsilon) \delta(p_i - \alpha) + \varepsilon \delta(p_i - \varepsilon^2) \\ q_i &= \frac{1}{2^{d-1}} - p_i \\ \rho(\pi_i^{(\mu)}) &= \delta(\pi_i^{(\mu)} - 1/2^d) \end{aligned} \quad (15)$$

where d is the dimension ; $\bar{P} = \varepsilon^2$, $\alpha > \frac{1}{2^d} + \varepsilon$ and $P_B = \varepsilon \ll \frac{1}{2^d}$ (note that $\varepsilon \rightarrow 0$) two arbitrary parameters, $\pi_i^{(\mu)}$ the probability of a hop to a direction μ different from the x -direction, and $i \equiv (i_1, \dots, i_d)$. As in this simple model there are no traps but only barriers we do not have to consider the sum $\sum^{(1)}$, of course. Moreover the « standard » random walk in the μ -directions can turn around the barriers.

Then the estimate (7) becomes in $d = 2$:

$$\bar{t}_1(|\gamma|) = \text{Min} \{ c\gamma^2, \bar{P}^{-1} \}, \quad (16)$$

and the corrections to the « naive » velocity will be :

$$V_x \gtrsim \langle p - q \rangle \left\{ 1 - 0\left(\frac{1}{|\ln P_B|^3}\right) - 0\left(\frac{P_B^{1/\sqrt{c\bar{P}}}}{\bar{P}}\right) \right\}. \quad (17)$$

They are always negligible as $\bar{P}^{1/\sqrt{c\bar{P}}} \rightarrow 1$ while $P_B \rightarrow 0$ when $P_B, \bar{P} \rightarrow 0$.

Now, we shall have for the distribution (15) the averages :

$$\langle p \rangle = \alpha - \varepsilon\alpha + \varepsilon^3; \quad \left\langle \frac{1}{p} \right\rangle = \frac{1 - \varepsilon}{\alpha} + \frac{1}{\varepsilon} \quad (18)$$

and therefore :

$$\langle p \rangle \simeq \alpha > \frac{1}{2^d}; \quad \left(\left\langle \frac{1}{p} \right\rangle \right)^{-1} \simeq \varepsilon. \quad (19)$$

It follows from the Derrida and Pomeau result [3] that in $d = 1$ the drift velocity can never be different from zero, just on the contrary of the $d = 2$ case where the velocity can never vanish !

As we have seen, the condition which assures a non zero velocity in a strip of width L is :

$$\bar{P}^{-1} P_B^L = \frac{\varepsilon^L}{\varepsilon^2} \ll 1. \quad (20)$$

Then a walk in a strip of width $L \geq 3$ has a non vanishing velocity as well as the walk in 2 dimensions.

Even if estimate (20) is not useful in the $L = 2$ strip, we suggest that the drift velocity may vanish in this case, according the value assumed by the average $\langle 1/p^2 \rangle$.

5. Conclusions.

We have analysed a model of non-symmetric random-random walk in two-dimensions and we have found that the region of zero drift velocity is either tiny or vanishing in the space of the parameters $P_B, \bar{P}$ of the particular class of probability distribution we consider provided the « naive » velocity $V^{(n)}$ is not exceedingly small. Namely the condition $\left\langle \left(\frac{1}{p} \right)^\alpha \right\rangle < 1, (\alpha \geq 1/6)$, is sufficient to assure a finite drift velocity V_x together with the further constraint $V_x^{(n)} > P_B^{1/2}$.

Moreover, we suggest that a transition to zero speed occur in a strip of width L at such values of $P_B, \bar{P}$ that $\langle 1/p^L \rangle \simeq 2$ (let us note $\left\langle \left(\frac{1}{p} \right)^\alpha \right\rangle \approx P_B \bar{P}^{-\alpha} + \langle p \rangle^{-\alpha}$ and $\langle p \rangle < 1$), even if we are able to give this rough estimate only for not too large L .

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