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Self-dual anisotropic two-dimensional Ising models with multispin interactions

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Résumé. — On introduit une famille de modèles d'Ising bidimensionnels anisotropes avec interactions à m spins dans une direction et on montre que ces modèles sont self-duaux. Ceci nous permet de localiser la ligne critique lorsque ces modèles présentent une transition de phase unique.

Abstract. — A class of anisotropic two-dimensional Ising models with m -spin interactions in one direction is introduced and shown to be self-dual. This enables us to locate the critical line when there is a unique phase transition.

In this letter we introduce a class of anisotropic two-dimensional (2D) Ising models with m -spin interactions in the spatial (horizontal) direction and the usual two-spin interaction in the temporal (vertical) direction on the rectangular lattice (Fig. 1). These models are self-dual and when unique, a phase transition occurs on the critical line

$$\sinh(2K_x) \sinh(2K_t) = 1 \quad (1)$$

where K_x and K_t are the couplings (in units of $k_B T$) for the spatial and the temporal directions.

Fig. 1. — Ising model with m -spin interactions K_x in the spatial direction and two-spin interactions K_t in the temporal direction ($m = 3$).

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The Hamiltonian of the system is

$$- \beta \mathcal{H}_m = K_\tau \sum_l s_i s_j + K_x \sum_L \prod_{k=1}^m s_k \quad (2)$$

where the $\{s\}$ are Ising spins ($s_i = \pm 1$); the first sum runs over the simple links l joining nearest-neighbours in the temporal direction; the second one is over the multiple links L joining m successive spins in the spatial direction. The rectangular lattice has N sites, N simple links l and N multiple links L ⁽¹⁾.

The partition function reads :

$$Z_m = \sum_{\{s\}} e^{-\beta \mathcal{H}_m} = \sum_{\{s\}} \prod_l \exp(K_\tau s_i s_j) \prod_L \exp\left(K_x \prod_{k=1}^m s_k\right). \quad (3)$$

Since $\prod_k s_k = \pm 1$ one may use the identity [1] :

$$\exp\left(K \prod_k s_k\right) \equiv \sum_{\lambda=0,1} C_\lambda(K) \left(\prod_k s_k\right)^\lambda \quad (4)$$

where :

$$C_0(K) = \cosh(K) \quad C_1(K) = \sinh(K) \quad (5)$$

to rewrite the partition function as :

$$Z_m = \sum_{\{s\}} \prod_l \sum_{\lambda_\tau=0,1} C_{\lambda_\tau}(K_\tau) (s_i s_j)^{\lambda_\tau} \prod_L \sum_{\lambda_x=0,1} C_{\lambda_x}(K_x) \left(\prod_{k=1}^m s_k\right)^{\lambda_x}. \quad (6)$$

Following Savit [1], we now rearrange the spin products in equation (6) so that all factors of s_i associated with a given site i are grouped together. Then :

$$Z_m = \sum_{\{\lambda\}} \prod_l C_{\lambda_\tau}(K_\tau) \prod_L C_{\lambda_x}(K_x) \sum_{\{s\}} \prod_i (s_i)^{\sum \lambda} \quad (7)$$

where the first sum is over the set of new variables λ_τ , $\lambda_x = 0,1$. There is one such variable for each link : one λ_τ for each l and one λ_x for each L . $\sum_i \lambda$ is the sum over the λ variables, two λ_τ and $m\lambda_x$, associated with site i (Fig. 2). Equation (7) may be rewritten as :

$$Z_m = \sum_{\{\lambda\}} \prod_l C_{\lambda_\tau}(K_\tau) \prod_L C_{\lambda_x}(K_x) \prod_i 2 \delta_2(\sum_i \lambda) \quad (8)$$

Fig. 2. — The two λ_τ and $m\lambda_x$ involving site i in the case $m = 3$.

⁽¹⁾ We take $K_\mu > 0$ ($\mu = x, \tau$) without loss of generality since replacing K_μ by $-K_\mu$ is equivalent to reversing the spins on some sublattice and leaves Z_m unchanged.

where $\delta_2(n)$ is a Kronecker delta function mod. 2, equal to one when n is even and zero when n is odd.

We will now look for a representation of the λ variables which automatically satisfies the Kronecker delta functions in equation (8) i.e. such that $\sum_i \lambda$ is even. To do this we introduce dual Ising variables $\sigma_i = \pm 1$ on the dual lattice (Fig. 3). When m is even the dual variables $\{ \sigma \}$

Fig. 3. — The original (heavy lines) and dual (dashed lines) lattices when m is even (a) or odd (b).

Fig. 4. — Correspondence between the λ_x and λ_z variables on the original lattice (heavy lines) and the spin products on the dual lattice (dashed lines) when m is even (a) or odd (b).

lie on sites situated in the centre of the plaquettes of the original lattice (i.e. the dual lattice is the same as for the usual Ising model), when m is odd they lie in the middle of the temporal links l of the original lattice. There is a one to one correspondence between simple (multiple) links l (L) and their associated variables $\lambda_\tau(\lambda_x)$ on the original lattice and multiple (simple) links L_d (l_d) and their associated dual variables $\prod_{k=1}^m \sigma_k(\sigma_i \sigma_j)$ on the dual lattice (Fig. 4) so that we can write :

$$\lambda_\tau(\sigma) = 1/2 \left(1 - \prod_{k=1}^m \sigma_k \right) \tag{9}$$

$$\lambda_x(\sigma) = 1/2 (1 - \sigma_i \sigma_j) . \tag{10}$$

In this representation, $\sum_i \lambda$ is automatically even. For example when $m = 3$ we get (Fig. 5) :

$$\sum_i \lambda = 5/2 - 1/2[\sigma_i \sigma_{i+\hat{\tau}} + \sigma_{i+\hat{x}} \sigma_{i+\hat{x}+\hat{\tau}} + \sigma_{i+2\hat{x}} \sigma_{i+2\hat{x}+\hat{\tau}} + \sigma_i \sigma_{i+\hat{x}} \sigma_{i+2\hat{x}} + \sigma_{i+\hat{\tau}} \sigma_{i+\hat{x}+\hat{\tau}} \sigma_{i+2\hat{x}+\hat{\tau}}]. \quad (11)$$

Fig. 5. — $\sum_i \lambda$ in the σ -representation (equations (9) and (10) in the text) when $m = 3$. $\hat{x}$ and $\hat{\tau}$ are unit vectors in the spatial and temporal directions.

Each spin enters two terms in the bracket and the sum of spin products may take on the values 5, 1, - 3. For any m , the possible values of the bracket are

$$m + 2 - 4v, \quad v \in N, \quad 0 \leq v \leq I(m/2) + 1 \quad (12)$$

(where $I(x)$ is the integer part of x) so that

$$\sum_i \lambda = 1/2[m + 2 - (m + 2 - 4v)] = 2v \quad (13)$$

is even.

It is clear from equations (9) and (10) that a number of $\{\sigma\}$ configurations lead to the same $\{\lambda\}$ (Fig. 6); in fact this number is equal to the ground-state degeneracy of the model (Fig. 7)

$$g = \sum_{i=0}^{I(m/2)} C_m^{2i} = 2^{m-1}. \quad (14)$$

Collecting these results together the partition function may be rewritten as

$$Z_m = 2^{1-m} 2^N \sum_{\{\sigma\}} \prod_{L_d} C_{\lambda_{\tau(\sigma)}}(K_{\tau}) \prod_{l_d} C_{\lambda_{x(\sigma)}}(K_x). \quad (15)$$

Making use of the identity :

$$C_{\lambda}(K) \equiv \cosh K \exp(\lambda \ln \tanh K) \quad (16)$$

we get :

$$Z_m = 2^{1-m} (2 \cosh K_{\tau} \sinh K_{\tau})^{N/2} (2 \cosh K_x \sinh K_x)^{N/2} \times \sum_{\{\sigma\}} \exp \left[\tilde{K}_{\tau} \sum_{l_d} \sigma_i \sigma_j + \tilde{K}_x \sum_{L_d} \prod_{k=1}^m \sigma_k \right] \quad (17)$$

or :

$$Z_m(K_{\tau}, K_x) = 2^{1-m} (\sinh 2 \tilde{K}_x \sinh 2 \tilde{K}_{\tau})^{-N/2} Z_m(\tilde{K}_{\tau}, \tilde{K}_x) \quad (18)$$

Fig. 6. — Two configurations $\{ \sigma \}$ on the dual lattice (dashed lines) leading to the same $\{ \lambda \}$ on the original lattice (heavy lines) for $m = 4$. The dual spin configuration on (b) is deduced from (a) by reversing all the spins along the columns indicated by arrows. The Hamiltonian of the system is left invariant under this global symmetry transformation.

Fig. 7. — The symmetry transformations leaving the 4-spin interaction invariant (reversed spins are indicated by dashed lines). When repeated periodically on the columns of the lattice as in figure 6, global symmetry transformations are generated.

where we introduced the dual couplings

$$\tilde{K}_\tau = -1/2 \ln \tanh K_x; \quad \tilde{K}_x = -1/2 \ln \tanh K_\tau \quad (19)$$

or

$$\sinh 2 \tilde{K}_\tau = (\sinh 2 K_x)^{-1}; \quad \sinh 2 \tilde{K}_x = (\sinh 2 K_\tau)^{-1}. \quad (20)$$

The free energy per site is

$$f_m(K_\tau, K_x) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_m(K_\tau, K_x) = -1/2 (\ln \sinh 2 \tilde{K}_\tau + \ln \sinh 2 \tilde{K}_x) + f_m(\tilde{K}_\tau, \tilde{K}_x) \quad (21)$$

and the model is self-dual for all m . In particular, the line

$$\sinh 2 K_x \sinh 2 K_\tau = 1 \quad (22)$$

which is invariant under the duality transformation given in equation (20), is the critical line of the model when there is a phase transition and when this transition is unique. Otherwise the duality transformation maps the low temperature region onto the high temperature region of the model.

When $m = 1$, we get noninteracting Ising chains in an external field $H = \beta^{-1} K_x$. This model has no transition but is known to be self-dual [2]. The model with $m = 2$ is the well-known 2D Ising model solved by Onsager [3] which was proved to be self-dual by Kramers and Wannier [4]. When $m = 3$ the model bears some similarity with the Baxter-Wu model [5], an Ising model with three-spin interactions on the triangular lattice. Both have the same ground-state degeneracy $g = 4$ (Fig. 8) and are related to a model with three-spin interactions on the rectangular lattice with alternate temporal interactions K_1 and K_2 . Our model is recovered when $K_1 = K_2 = K_\tau$.

Fig. 8. — Triangular lattice with 3-spin interactions. (a) The four degenerate ground-states of the elementary interaction. (b) One of the four corresponding global symmetry operations (reversed spins (squares) lie on an hexagonal sublattice).

and the Baxter-Wu model corresponds to the limit $K_1 \rightarrow \infty, K_2 \rightarrow 0$ (Fig. 9). We suspect these models to belong to the same class of universality which might be also that of the 4-state Potts model [6] according to the Den Nijs conjecture [7]. The transition probably becomes first-order above $m = 3$ as indicated by mean-field theory on a 1D quantum formulation of the model.

(a)

(b)

Fig. 9. — Generalized Ising model with 3-spin interactions (a) and the corresponding Baxter-Wu model (b).

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