Ring polymers in solution: topological effects
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1. Introduction. — The static properties of linear polymers in good solvents depend on excluded volume interactions and are now rather well known. The properties of ring polymers in good solvents depend not only on the same kind of excluded volume interactions but also on topological constraints. In spite of a few studies [1, 2] by computer simulation, the effects of these constraints remain almost unknown [3]. We expect now that good samples of long ring polymers will soon be available [4]. Thus, it seems worthwhile to reexamine this interesting but difficult question.

It is clear that short rigid rings and long flexible rings have very different properties; therefore we shall study successively these two limiting cases, the second one appearing especially interesting.

2. Short rigid rings in solutions. — Short rigid rings in solutions have nearly a circular shape and can be considered in a first approximation as simple molecules. Their properties come from the fact that the rings cannot penetrate each other. The size of a ring can be defined by the radius of gyration $R_g$, and the mean volume occupied by a polymer ring we define to be

$$V_G = \frac{4}{3} \pi R_g^3. \quad (2.1)$$

The effect of the constraints can be observed by measuring the osmotic pressure $\Pi$. This quantity can be expressed in terms of the dimensionless parameter $CV_G$, where $C$ is the number of polymers per unit volume and the coefficients of the expansion of $\Pi$ with respect to $CV_G$ are pure numbers.

$$\Pi \beta = C \left[ 1 + \frac{1}{2} GV_G + \cdots \right]. \quad (2.2)$$

Here

$$G = V/V_G. \quad (2.3)$$

where $V$ is the excluded volume.

For circles of radius $R$, we found

$$R_G = R \quad V_G = 4\pi R^3/3 \quad V = 8\pi R^3/3$$

and therefore

$$G = 2. \quad (2.4)$$

For comparison, we remark that a similar calculation for non intersecting discs of radius $R$ gave us

$$R_G = 2^{-1/2} R \quad V_G = 2^{1/2} \frac{\pi}{3} R^3$$

$$V = \left(\frac{\pi}{2} + \frac{4}{3}\right)\pi R^3$$

and therefore

$$G = 2^{-3/2}(3\pi + 8) = 6.16 \quad (2.5)$$
For solid hard balls of radius $R$, we would get

$$R_0 = (3/5)^{1/2} R \quad V_0 = \frac{4\pi}{5} (3/5)^{3/2} R^3$$

and

$$G = 8(5/3)^{3/2} = 17.21.$$ (2.6)

Further terms of the expansion of the osmotic pressure could also be calculated but the calculation would be more complicated. In particular, it would be interesting to consider the case where $CR^d$ is large. Then the rings would form layers of rings parallel to one another (nematic configuration).

It would also be possible to consider the case of slightly flexible rings and to calculate by perturbation the effects of this small flexibility.

All these calculations may be complicated but the basic ideas are simple. Thus, we feel that we understand and can control this limiting case rather well.

3. Long flexible rings. — It is much more difficult to study the effect of topological constraints on the properties of long flexible ring polymers in a solvent. For simplicity we shall assume that the long flexible rings have the same topology as a circle (trivial knot). In this way, the problem is well defined. We must exclude all configurations in which two (or several) ring polymers are linked (i.e., knotted configurations); this means that a configuration is excluded when the rings cannot be separated from one another by continuous deformation without crossing. Unfortunately, there are no simple classifications of knots and no simple ways of determining whether or not two rings make a knot.

However, some information can be obtained by studying the linking number of two rings: $C_a$ and $C_b$. This simple topological invariant is the integer given by Gauss’ formula [5]:

$$I_{ab} = \frac{1}{4\pi} \oint_{C_a} \oint_{C_b} \frac{(dr_a \wedge dr_b) \cdot r_{ab}}{r_{ab}^3}.$$ (3.1)

If $C_a$ and $C_b$ are not linked, the linking number is zero; consequently if the linking number of $C_a$ and $C_b$ is not zero, $C_a$ and $C_b$ are linked.

The reverse is less conclusive; in general, if the linking number is zero, $C_a$ and $C_b$ may be linked or not linked; however they are not linked if $C_a$ and $C_b$ are plane convex curves.

Thus the problem could be simplified by making the approximate assumptions that the excluded configurations are those for which the linking number is not zero. Thus it is useful to obtain information concerning the linking numbers of random flexible rings $C_a$ and $C_b$.

Let $\rho$ be the vector joining the centre of gravity of $C_a$ to the centre of gravity of $C_b$. Let us fix $\rho$ and let us assume that the orientations and the shapes of $C_a$ and $C_b$ are at random. By averaging over these orientations and shapes, we define the mean square linking number

$$J(\rho) = \langle [I_{ab} (\rho)]^2 \rangle$$

and the volume

$$J = \int d^3\rho J(\rho).$$ (3.2)

Incidentally, we note that, if $C_a$ and $C_b$ were plane convex curves, we would obtain

$$J = V$$

where $V$ is the average excluded volume of the curves, but in general $J$ and $V$ are different quantities.

It happens that $J$ can be expressed in a very simple way as was shown recently by W. Pohl [6] and also by J. des Cloizeaux and R. Ball [7] and by B. Duplantier [8]. Consider random curves $C_a$ and $C_b$. $J$ is given by [7]

$$J = \frac{1}{8\pi} \oint_0^\infty dr A_a (r) A_b (r)$$ (3.3)

where $A_a (r)$ and $A_b (r)$ are characteristic functions of $C_a$ and $C_b$. The function $A(r)$ associated with a random closed curve $C$ is defined by [6]

$$A(r) = \frac{1}{r} \int_C \int_C \frac{dr_1 \cdot dr_2 \cdot \delta (r - r_12)}{r_1 r_2}$$

$$= \frac{1}{r^2} \int_C \int_C (dr_1 \cdot r_12) (dr_2 \cdot r_12) \cdot \delta (r - r_12)$$ (3.4)

where $r_1$ and $r_2$ define respectively the positions of points $M_1$ and $M_2$ on $C(\theta(x)$ is the step function). The integrals are taken along $C$.

The function $A(r)$ for a long flexible ring of length $L$ can be approximated by an expression of the form

$$A(r) = L \phi (r/\lambda)$$ (3.5)

where $\lambda$ is a persistence length. Using (3.4), we verify that $\phi (0) = 2$ and that $\phi (x)$ goes to zero when $x$ becomes large. Thus we may write

$$J \approx \frac{L_a L_b}{8\pi} \oint_0^\infty dr \varphi (r/\lambda)$$ (3.6)

and we claim that, in general, the integral is convergent. Thus we obtain

$$J \approx K L_1 L_2 \lambda$$ (3.7)

which is valid for large values of $L_1/\lambda$ and $L_2/\lambda$. Here $K$ is a constant which depends on the nature of the random curves.

The fact that the preceding integral (3.6) converges is of course crucial and can be shown in different ways.
For instance, we calculated $K$ for a long gaussian ring. In this case, the persistence length $\lambda$ can be defined by setting

$$\left\langle \frac{\text{d}r_1}{\text{d}t} \cdot \frac{\text{d}r_2}{\text{d}t} \right\rangle = \exp[-|l_1 - l_2|/\lambda]$$

and we found

$$\frac{|l_1 - l_2|}{L} \ll 1$$

(3.8)

and therefore

$$\langle n \rangle = J(\rho) .$$

(3.11)

By combining equations (3.10) and (3.11), we come to the conclusion that the total number $n$ of loops of $C_a$ around $C_b$ is proportional to the number of contacts of these curves.

We can now generalize this idea. We may assume that two pieces of curve make knots when they come close to each other and that otherwise the probability of making knots is rather low and unimportant.

Thus, we are led to the following conjecture: long ring polymers are subjected to topological constraints but these constraints produce only excluded volume effects.

According to this conjecture, the critical indices of ring polymers in good solvents should be the same as those of linear polymers in good solvents. However, ring polymers should be more soluble than linear polymers and their Flory temperatures ($\theta$ point) should not exactly coincide.

The quantity $b$ which defines the interaction in the two parameter model has a dimension [9]

$$b = (\text{length})^{d-1} \quad (d = \text{space dimension})$$

Thus for ring polymers in solution the part of the interaction which results from topological constraints should be proportional to $\lambda$.

We may write

$$b \sim K\lambda .$$

However to find more precise values of $b$ a specific model has to be chosen and elaborate calculations are required.

Finally, we note that the results presented in a recent letter by Brereton and Shah [10] are compatible with ours.

4. Conclusion. — The main result of this discussion is the conjecture that for long flexible rings, the topological constraints and the excluded volume interactions have similar effects. This idea has now to be tested by conducting experiments on long ring polymers. We also have to perform more sophisticated calculations and to see whether the results obtained agree with the preceding simple conjecture.

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References


[7] Des Cloizeaux, J. and Ball, R., To be published in Communications in Mathematical Physics.
[8] Duplantier, B., Submitted for publication to Communications in Mathematical Physics.