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Anderson localization, branched polymers and the Yang-Lee edge singularity

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Résumé. — On montre qu'un modèle de champs récemment proposé pour la localisation d'Anderson a les mêmes propriétés critiques que celles du problème des polymères branchés. Ce modèle dans D dimensions appartient donc à la même classe d'universalité que celle étudiée par Parisi et Sourlas qui inclut la singularité de Yang-Lee à $D - 2$ dimensions. On montre qu'on peut étudier les propriétés critiques à paramètre d'ordre constant ou à champ constant et que les exposants critiques dans les deux cas sont reliés par une renormalisation de Fisher. On note que les exposants à paramètre d'ordre constant divergent pour une dimension critique D_c .

Abstract. — We show that a recently proposed field-theoretic model of Anderson localization has the same critical behaviour as that found in the problem of branched polymers. Thus this model in D dimensions is in the same universality class as that studied by Parisi and Sourlas, which includes the Yang-Lee edge singularity in $D - 2$ dimensions. We stress that critical properties can be studied either at constant order parameter or at constant field and that critical exponents in the two cases are related by a Fisher renormalization. We note that constant order parameter exponents diverge at a critical dimension D_c .

Recently Parisi and Sourlas [1] have shown that the critical exponents arising in the study of branched polymers in good solvents [2] in D dimensions are related to those that describe the Yang-Lee edge singularity [3] in $d = D - 2$ dimensions. Their method consisted of showing that, in the critical region, the dominant Feynman diagrams appearing in the field-theoretic formulation of the branched polymer problem were exactly those that appear in a ϕ^3 field theory with a quenched random imaginary magnetic field. The connection between the critical behaviour of systems in quenched random magnetic fields in D dimensions and the corresponding system without randomness in $D - 2$ dimensions [4] was then used to prove the result (the fact that the magnetic field is imaginary just leads to the coupling constant of the ϕ^3 interaction being imaginary, as is required for the Yang-Lee problem).

In this note we apply the same ideas to a field-theoretic model of Anderson localization [5]. This model, although more complex than that of the branched polymers, has many features in common

with it. In particular, the critical exponents for the two problems have been shown to be the same to lowest order in an ε -expansion about eight dimensions. We will show that this is an exact equality, within the ε -expansion, and so this problem falls into the same universality class investigated by Parisi and Sourlas, for sufficiently large D . In discussing the localization problem, we will also make some general comments about critical exponents for the branched polymer and Yang-Lee problem. In particular, we reemphasize that in all three problems there is only one independent critical exponent from which all others can be obtained, and point out that the Flory approximation for the correlation length exponent, ν^* , in the polymer problem leads to exponents in extremely close agreement in all dimensions with those calculated by other methods. We also point out that in all three problems, quantities can be investigated either at constant order parameter Q or external field H , and that exponents for the latter case are related to those of the former by a conventional Fisher renormalization [6].

The model we wish to study has the Hamiltonian

$$H = \int d^d x \left\{ \frac{1}{2} \text{Tr} (\nabla Q \nabla Q) + \frac{1}{2} \sum_{p, p' = +, -} r_{pp'} \text{Tr} (Q_{pp'} Q_{pp'}^T) + \frac{1}{2} \sum_{p, p' = +, -} \tau_{pp'} (\text{Tr} Q_{pp}) (\text{Tr} Q_{p'p'}) + \frac{w}{3!} \text{Tr} (Q^3) \right\} \quad (1)$$

where Q is a symmetric $2n \times 2n$ matrix with four $n \times n$ submatrices Q_{++} , Q_{+-} , Q_{-+} and Q_{--} :

$$Q = \begin{bmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{bmatrix}. \quad (2)$$

This model was obtained using the replica trick, and so it is understood that n is to be set equal to zero at the end of any calculation. The $n \rightarrow 0$ nature

of the problem together with the unconventional third term in equation (1) implies that the upper critical dimension of the model is eight and not six (see reference [5]. The possibility that eight and not six was the upper critical dimension was also noted by Parisi [7]).

Let us first look at the Hamiltonian (1) keeping fields in the $++$ subspace only. As we will see, the study of the critical behaviour simplifies considerably if one introduces fields φ_α and $\psi_{\alpha\beta}$ ($\alpha, \beta = 1, \dots, n$; $\alpha < \beta$) by

$$Q_{++} = \begin{bmatrix} \varphi_1 & & & 0 \\ & \varphi_2 & & \\ & & \ddots & \\ 0 & & & \varphi_n \end{bmatrix} + \begin{bmatrix} 0 & \psi_{12} & \psi_{13} & \dots & \psi_{1n} \\ \psi_{12} & 0 & \psi_{23} & \dots & \psi_{2n} \\ \vdots & & \vdots & & \vdots \\ \psi_{1n} & \psi_{2n} & \psi_{3n} & \dots & 0 \end{bmatrix} \quad (3)$$

where we will refer to the second matrix on the right-hand-side of equation (3) simply as ψ . The Hamiltonian for these fields in the $++$ subspace then becomes

$$H(++) = \int d^d x \left\{ \frac{1}{2} \sum_{\alpha=1}^n [(\nabla \varphi_\alpha)^2 + r_{++} \varphi_\alpha^2] + \frac{1}{2} [\text{Tr}(\nabla \psi)^2 + r_{++} \text{Tr}(\psi^2)] + \right. \\ \left. + \frac{1}{2} \tau_{++} \left(\sum_{\alpha=1}^n \varphi_\alpha \right)^2 + \frac{w}{3!} \sum_{\alpha=1}^n \varphi_\alpha^3 + \frac{w}{2} \sum_{\alpha \neq \beta}^n \varphi_\alpha \varphi_{\alpha\beta}^2 + \frac{w}{3!} \text{Tr}(\psi^3) \right\}. \quad (4)$$

Thus we see that while the ψ fields have the usual $(q^2 + r)^{-1}$ propagator, the φ fields have a propagator

$$P_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{(q^2 + r)} - \frac{\tau}{(q^2 + r)(q^2 + r + n\tau)} \quad (5)$$

exactly as found by Parisi and Sourlas in the branched polymer problem [1]. Denoting the first term on the right-hand-side of equation (5) by a solid line and the second term by a dotted line, we may analyse the Feynman diagrams arising from (4) in the same way as they did. We work only with $\langle \varphi_\alpha \varphi_\beta \rangle$ and $\langle \varphi_\alpha \varphi_\beta \varphi_\gamma \rangle$ (proportional to $\delta_{\alpha\beta}$ and $\delta_{\alpha\beta\gamma}$ respectively) since by the symmetry of the original Hamiltonian in Q_{++} , the other Green's functions such as $\langle \psi \psi \rangle$ or $\langle \varphi \psi \psi \rangle$, $\langle \psi \psi \psi \rangle$ are equal up to trivial normalization. Clearly the most important diagrams in the critical region will be those with the most dotted lines, since these will be the most infrared divergent. However, many of these do not contribute since

a) they are not « connected », in the sense that the external legs are not connected to each other by solid lines and so the diagram is not proportional to $\delta_{\alpha\beta}$ or $\delta_{\alpha\beta\gamma}$;

b) there is an isolated « cluster » of solid lines among dotted lines. This leads to a free sum on an index, since the « cluster » can be in any one of n « states », and so gives zero when $n \rightarrow 0$.

As noted by Parisi and Sourlas these are exactly the same rules which appear in the graphical analysis of the percolation problem [8]. These and other similar considerations [4] show that the most dominant l -loop diagrams that contribute are contained in the set of diagrams with l dotted lines. However, we find that in our model (4) all these diagrams have $\sum_\alpha \varphi_\alpha^3$ vertices only, and the two interactions in (4) involving the ψ fields never appear. Thus the effective Hamiltonian in the critical region is just

$$H(++) = \int d^d x \left\{ \frac{1}{2} \sum_{\alpha=1}^n [(\nabla \varphi_\alpha)^2 + r_{++} \varphi_\alpha^2] + \frac{1}{2} \tau_{++} \left(\sum_{\alpha=1}^n \varphi_\alpha \right)^2 + \frac{w}{3!} \sum_{\alpha=1}^n \varphi_\alpha^3 \right\} \quad (6)$$

which is exactly the model considered by Parisi and Sourlas. Thus we have shown that, in the $++$ sector, the model of Anderson localization (1) has the same critical properties as that of branched polymers, and therefore of the Yang-Lee edge problem in two less dimensions.

Now let us go on to investigate the full model (1) in the same way. We see the possibility of a richer structure emerging here than was possible previously, due to the existence of the various sectors which may not behave symmetrically. However, at the fixed point

in $D = 8 - \varepsilon$ dimensions $r_{++} = r_{+-}^0 = r_{--}$ and $\tau_{++} = \tau_{+-} = \tau_{--}$ [5], and so our analysis goes through as before. We may therefore conclude that the model of Anderson localization (1) is in the same universality class as the branched polymer problem for D sufficiently large. In lower dimensions we expect that this symmetry between the sectors will no longer hold and that consequently it is probable that there is a crossover to a new fixed point as D decreases.

When discussing the critical exponents in the Yang-Lee, branched polymer and localization problems, we will take care to distinguish between the « constant Q » and « constant H » exponents by attaching an asterisk to the constant H ones. The constant Q exponents are the natural ones that arise when working with a field theory; of course the exponent η is the same in both schemes since it is defined right at the critical point. We will also distinguish between the Yang-Lee exponents and the exponents for the branched polymer problem and localization, by putting the subscript YL on the former.

The critical exponents v_{YL} and v_{YL}^* are not independent of η_{YL} . Fisher [3] has shown that hyperscaling arguments lead to $v_{YL}^{*-1} = \frac{1}{2}(d + 2 - \eta_{YL})$. Further, as pointed out in reference [9], in the Yang-Lee case the anomalous dimension of the ϕ^2 operator is linearly related to the anomalous dimension of the ϕ operator and the β -function, from which it follows that $v_{YL}^{-1} = \frac{1}{2}(d - 2 + \eta_{YL})$. The exponent that characterizes the singularities, called σ by Fisher [3], is in our notation β_{YL}^* and is given by

$$\beta_{YL}^* = \frac{d - 2 + \eta_{YL}}{d + 2 - \eta_{YL}} \quad (7)$$

so that

$$\beta_{YL}^* = \frac{1}{v_{YL} d - 1}, \quad v_{YL}^* = \frac{v_{YL}}{v_{YL} d - 1}. \quad (8)$$

The relationship between this model and that of branched polymers or localization shows that η and v in D dimensions equal η_{YL} and v_{YL} respectively in $d = D - 2$ dimensions. Hyperscaling in the latter models is violated in the usual simple way for systems in a random magnetic field : D is replaced by $D - 2$ in the scaling laws. One therefore finds the following results for the constant Q exponents in the branched polymer problem

$$v^{-1} = \frac{1}{2}((D - 2) - 2 + \eta), \quad \gamma = (2 - \eta) v, \quad v(D - 2) = 2 - \alpha \quad (9)$$

which imply

$$(D - 2) v = 2 + \gamma, \quad \gamma = -\alpha, \quad \beta^* = \frac{1}{1 + \gamma}. \quad (10)$$

However, it is known [2] that the constant Q and constant H exponents for branched polymers are related by

$$\beta^* = \frac{1}{2\gamma - \mu_3}, \quad \gamma^* = \frac{\gamma}{2\gamma - \mu_3}, \quad v^* = \frac{v}{2\gamma - \mu_3} \quad (11)$$

where μ_3 is a critical exponent controlling the critical behaviour of an irrelevant three point vertex (note that in reference [2] the constant H exponents were denoted by a subscript « a » for « animals », since the branched polymer problem is in the same universality class as lattice animals). One sees from equations (10) and (11) that μ_3 is not an independent exponent : $\mu_3 = \gamma - 1$. Equations (9)-(11) now give

$$v^*(D - 2) = 2 - \alpha^*, \quad \beta^* = \frac{1}{2} v^*((D - 2) - 2 + \eta), \quad \gamma^* = (2 - \eta) v^*, \quad \beta^* + \gamma^* = 1 \quad (12)$$

with

$$\beta^* = \frac{1}{1 - \alpha}, \quad \gamma^* = \frac{\gamma}{1 - \alpha}, \quad \alpha^* = \frac{-\alpha}{1 - \alpha}, \quad v^* = \frac{v}{1 - \alpha}. \quad (13)$$

Thus the constant H exponents are related to those at constant Q by a conventional Fisher renormalization [6]. One also sees from equation (12) that the constant H exponents violate hyperscaling in the same simple way as the constant Q exponents, and also satisfy an additional relation : $\beta^* + \gamma^* = 1$.

In order to obtain reliable estimates for these exponents we will use the results from the Yang-Lee case rather than those from the branched polymers, since they are more accurately known. Estimates have been obtained from both high temperature series analysis [3] and the ε -expansion [9]. A detailed analysis of the ε -expansion results shows impressive agreement between the two approaches [10]. We note that a good phenomenological fit to η_{YL} for $6 \geq d \geq 1$ is $\eta_{YL} = -\frac{1}{5}(6 - d)$ (Fig. 4, Ref. [10]) which leads to $\eta = \frac{1}{5}(D - 8)$ for $8 \geq D \geq 3$. Using equation (12) we find that $v^* = 5/2(D + 2)$, which is just the Flory formula in the branched polymer case [11]. The structure of the exponents is well brought out by using the Flory result for v^* . One finds in this approximation

$$\beta^* = \frac{3D - 14}{2(D + 2)}, \quad \gamma^* = \frac{18 - D}{2(D + 2)}, \quad v^* = \frac{5}{2(D + 2)}, \quad \eta = \frac{D - 8}{5}, \quad \gamma = \frac{18 - D}{3D - 14}, \quad v = \frac{5}{3D - 14}. \quad (14)$$

These formulae can be used for all $D \leq 8$ and in fact give the exact results for $D = 2$ and 3 . We see that there exists a critical dimension D_c (which equals $14/3$ in the Flory approximation) at which the constant Q exponents become infinite. The constant H exponents, however, remain well defined, with $\beta^* < 0$ for $D < D_c$. We can now go on to discuss the consequences of equations (12)-(14) for the localization problem.

An analysis of the model (1) led to the following behaviour for the density of states, $\rho(E)$, and the conductivity, $\Sigma(E)$, near the mobility edge [5] :

$$\begin{aligned} \rho_{\text{loc.}}(E) &\sim \exp(-\text{const.}(\Delta E)^{-\zeta}) \\ \rho_{\text{ext.}}(E) &\sim (\Delta E)^{\beta^*} \\ \Sigma(E) &\sim (\Delta E)^{\mu} \end{aligned} \quad (15)$$

The subscripts loc. and ext. refer to localized states and extended states respectively and $\Delta E = |E - E_c|$ where E_c is the energy at the mobility edge. The exponents ζ and μ are given in terms of the constant H exponents,

$$\zeta = (D - 2 + \eta) v^* - 2 \beta^*$$

and $\mu = 2 - \gamma^* - 2 v^*$ for $D < 8$. Using equation (12) we can simplify these expressions to yield

$\zeta = 2 v^*$ and $\mu = (D - 4) v^*$. Thus we have for the localization exponents

$$\mu = \begin{cases} 1, & D > 8 \\ (D - 4) v^*, & D < 8 \end{cases} \quad (16)$$

$$\beta^* = \begin{cases} \frac{1}{2}, & D > 8 \\ (D - 2) v^* - 1, & D < 8 \end{cases} \quad (17)$$

$$\zeta = \begin{cases} \frac{1}{4}(D - 6), & D > 8 \\ 2 v^*, & D < 8. \end{cases} \quad (18)$$

In the branched polymer case the exponents given by (12) and (14) seem to fit the data well for the whole range of dimensionalities from two to eight. We do not expect this to be the case with the localization exponents (16)-(18) since it has recently become clear that the form for the density of states $\rho(E)$ given by (15) violate theorems [12] which state that, for a wide class of models including the one we have considered here, $\rho(E)$ should have no zeros for real E . It is not clear where the analysis leading to the results (15) breaks down. It may be that this behaviour is only found near some value of E_c which is complex, and therefore unphysical, having little influence on the behaviour of $\rho(E)$ for real E [7].

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