Exact results for a two-dimensional one-component plasma on a sphere
J.M. Caillol

To cite this version:
J.M. Caillol. Exact results for a two-dimensional one-component plasma on a sphere. Journal de Physique Lettres, 1981, 42 (12), pp.245-247. <10.1051/jphyslet:019810042012024500>. <jpa-00231919>

HAL Id: jpa-00231919
https://hal.archives-ouvertes.fr/jpa-00231919
Submitted on 1 Jan 1981

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Exact results for a two-dimensional one-component plasma on a sphere

J. M. Caillol

Laboratoire de Physique Théorique et Hautes Energies (*), Université de Paris-Sud, 91405 Orsay, France

(Reçu le 20 mars 1981, accepté le 29 avril 1981)

Résumé. — Pour étudier le modèle OCP du plasma classique à deux dimensions, nous proposons de placer les particules à la surface d’une sphère. Pour une valeur particulière de la température $T_o$, la fonction de partition ainsi que les fonctions de distribution peuvent être calculées explicitement. A cette température la limite thermodynamique de ce système coïncide avec celle du système plan.

Abstract. — In order to study the properties of the two-dimensional classical one-component plasma, we propose to confine the particles to the surface of a sphere. For a special value $T_o$ of the temperature, the partition and the distribution function can be calculated explicitly. The thermodynamic limit of this system coincides, for $T = T_o$, with the one obtained previously for a planar system.

In two recent articles Jancovici and Alastuey (JA) [1, 2] studied the equilibrium statistical mechanics of a classical two-dimensional one-component plasma (OCP) using methods of random matrices [3, 4]. For some special value $T_o$ of the temperature they calculated the free energy and the $n$-body distribution functions exactly.

In this letter we consider the case of a system of $N$ particles confined to the surface of a sphere. We show that for the temperature $T_o$ this system has the same thermodynamic limit as the one considered by (JA).

The $N$ particles of mass $m$ and charge $q$ are confined to the surface of a sphere centred in $0$ and of radius $R$. The number density is

$$\rho = \frac{N}{4\pi R^2}.\quad (1)$$

The interaction potential between particles $i$ and $j$ located at $M_i$ and $M_j$ on the sphere is

$$v_{ij} = - \frac{\alpha^2}{2} \log \left( \frac{r_{ij}}{L} \right).\quad (2)$$

where $L$ is a length scale and $r_{ij}$ is the length of the chord joining $M_i$ and $M_j$, i.e.

$$r_{ij} = 2R \sin \left( \frac{\psi_{ij}}{2} \right).$$

with

$$\psi_{ij} = \arccos \left( \frac{OM_i \cdot OM_j}{R^2} \right).\quad (4)$$

The particles are embedded in a uniform neutralizing background of opposite charge.

The total potential energy, taking into account the influence of the background, is

$$V_N(1, 2, ..., N) = - \frac{\alpha^2}{2} \sum_{i=1}^{N} \sum_{j>i}^{N} \log \frac{2R^2}{L^2} \times$$

$$x (1 - \cos \psi_{ij}) - \frac{N^2 \alpha^2}{4} \left( 1 - \log \frac{4R^2}{L^2} \right).\quad (5)$$

1. Excess free energy. — For the temperature $T_o = q^2/2k_B$ ($k_B$ Boltzmann constant) the excess canonical partition function is

$$Z_N(T_o) = e^{N^2/2} \cdot \left( \frac{L}{2} \right)^N \cdot R^N \int \prod_{i=1}^{N} d\Omega_i \times$$

$$\times \prod_{k=1}^{N} \prod_{l > k} \left( 1 - \cos \psi_{kl} \right)$$

where

$$d\Omega_i = \sin \theta_i \, d\theta_i \, d\varphi_i$$

($\theta_i$, $\varphi_i$ spherical coordinates of $M_i$).
Introducing the Cayley-Klein parameters defined by

\[ \alpha_i = \cos \frac{\theta_i}{2} e^{i \phi_i / 2} \]
\[ \beta_i = -i \sin \frac{\theta_i}{2} e^{-i \phi_i / 2} \]  

we can write

\[ 1 - \cos \psi_{ij} = 2 | \alpha_i \beta_j - \alpha_j \beta_i |^2. \]  

The integrand of (6) now takes the form

\[ \frac{1 - \cos \psi_{ij}}{2} = \prod_{k=1}^{N} \prod_{j>i}^{N} \left( \frac{\alpha_i - \alpha_j}{\beta_i - \beta_j} \right). \]  

The second product in the r.h.s. of equation (11) is a Vandermonde determinant. Expanding it and inserting into equation (6) we get

\[ Z_N(T_0) = e^{\frac{1}{2}} (2 \pi L)^N R^N N! \times \]
\[ \times \prod_{k=1}^{N} \frac{(k-1)! (N-k)!}{N!}. \]  

This result has a structure similar to the expression derived by (JA) for a charged disk of radius R

\[ Z_N(T_0) = K \pi^N N! \prod_{k=1}^{N} \gamma(k,N) \]  

where \( \gamma(k,N) \) is the incomplete gamma function [2], and K is a constant.

The difference is that our calculation involves complete gamma functions.

The excess free energy for \( T = T_0 \) is readily derived from (12)

\[ F_N(T_0) = -k_B T_0 \log \frac{Z_N(T_0)}{\pi R^N N!}. \]  

In the thermodynamic limit (\( N \to \infty, R \to \infty, \rho \) constant) one obtains

\[ \lim_{N \to \infty} \frac{F_N(T_0)}{N} = q^2 \left[ \frac{1}{2} - \frac{1}{4} \log 2 \pi \right] - \]
\[ - \frac{q^2}{4} \log (\rho \pi L^2) + O \left( \frac{\log N}{N} \right) \]  

which is identical to the result given in reference [1].

2. Distribution functions. — The n-particle distribution function for \( T = T_0 \) is given by

\[ \rho_{N,T_0}^{(n)}(1,2,\ldots,n) = \frac{N!}{(N-n)!} \frac{1}{Z_N(T_0)} \times \]
\[ \times \int \prod_{k=n+1}^{N} d\Omega_k \prod_{i=1}^{N} \prod_{j>i}^{N} \left( \frac{1 - \cos \psi_{ij}}{2} \right). \]  

Using arguments parallel to those given in reference [2] one finds

\[ \rho_{N,T_0}^{(1)}(1) = \rho \]
\[ \rho_{N,T_0}^{(2)}(1,2) = \rho^2 \]  

with

\[ 1 \leq k \leq n \]
\[ 1 \leq l \leq n. \]  

In particular

\[ \rho_{N,T_0}^{(1)}(1) = \rho \]
\[ \rho_{N,T_0}^{(2)}(1,2) = \rho^2 \]  

One can note that for \( \psi_{12} = \pi, \rho_{N,T_0}(12) = 1. \)

The system appears to be homogeneous for all \( N \) and the distribution functions are invariant by rotation of the sphere. Indeed, under a rotation \( R_0 \) characterized by its Cayley-Klein parameters, the coordinates \( (\alpha_i, \beta_i) \) of particle \( i \) transform in the following way

\[ \left( \frac{\alpha_i}{\beta_i} \right) = \left( \frac{\alpha_0}{\beta_0} \right) \left( - \beta_0^* \cdot \alpha_0^* \right). \]  

Using the fact that

\[ \alpha_0 \beta_0^* + \beta_0 \alpha_0^* = 1 \]  

and

\[ \alpha_i \beta_i^* + \beta_i \alpha_i^* = 1 \]  

we conclude that

\[ \alpha_i' \beta_i'^* + \beta_i' \alpha_i'^* = \alpha_i \beta_i^* + \beta_i \alpha_i^* \]  

and consequently \( \rho_{N,T_0}^{(1)}(1,\ldots,n) \) is invariant under the rotation \( R_0 \). In particular expression (20) shows that \( \rho_{N,T_0}^{(2)}(12) \) depends only on the relative distance between particles 1 and 2.

The thermodynamic limit of the functions \( \rho_{N,T_0}^{(n)}(1,\ldots,n) \) is obtained in the following way : Define \( \rho_k = R \theta_k \). The limit \( N \to \infty, R \to \infty, \rho \) constant, is taken by keeping \( \rho_k \) and \( \phi_k \) constant for each particle \( k (1 \leq k \leq n) \). For an infinitely large sphere the particles will be situated in the tangent plane at the North Pole and there positions will be characterized by the polar coordinates \( (\rho_k, \phi_k) \).

An expansion of (17) to second order in \( \rho_k / R \) gives

\[ \rho_{N,T_0}^{(n)}(1,2,\ldots,n) = \rho^* \exp \left[ - \sum_{k=1}^{n} \frac{1}{2} | Z_k |^2 \right] \times \]
\[ \times \det \left[ \exp(Z_k^* Z_l) \right] \]  

where \( Z_k \) is the complex coordinate of the particle k at the North Pole,

\[ Z_k = \rho_k e^{i \phi_k}. \]
where $Z_k$ is defined by

$$Z_k = \sqrt{\pi \rho_k e^{i\phi_k}}. \quad (25)$$

The expression (24) coincides with the thermodynamic limit of the $n$-particle distribution function of a planar system.

3. Conclusion. — The thermodynamic limit of the system we have considered is identical to that of the planar two-dimensional OCP. It seems likely that this result remains valid for any temperature.

Acknowledgments. — I thank A. Alastuey, B. Jancovici, D. Levesque, and J. J. Weis for stimulating discussions.

References