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Asymmetrical loading of a penny-shaped crack in an infinite transversely isotropic medium (*)

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Résumé. — La distribution des contraintes et des déplacements dans un massif infini transversalement isotope, contenant une fissure plane circulaire, est déterminée quand des forces intérieures, concentrées sur une surface, sont appliquées d’un seul côté de la fissure, à une distance finie.

Abstract. — The problem of stress and displacement fields in an infinite transversely isotropic medium is solved when a penny-shaped crack is opened by an asymmetrical system of body forces; the body forces are concentrated on one interior surface situated at a finite distance from the crack.

1. Introduction. — The problem of stress and displacement fields in an infinite medium in the presence of a symmetrical system of body forces has been solved by Sneddon and Tweed [1] for an isotropic material and by Dahan [2] for a transversely isotropic material. The problem of crack opening by an asymmetrical system of body forces, acting on one side of the crack, has been considered by Collins [3] in the isotropic case. The purpose of this paper is to give the complete solution of this problem when the medium is characterized by transverse isotropy, having the $z$-axis as the elastic symmetric axis. We shall assume that the loading is concentrated on a plane internal disc, around the $z$-axis, situated at a finite distance $h$ from the crack.

By using the cylindrical coordinates $(r, \theta, z)$, we shall note $(u_r, u_\theta, u_z)$ the components of the displacement field $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz})$ the non-zero components of the stress tensor. The penny-shaped crack is defined by its radius $r_0$ and it is situated in the plane $z = 0$ (cf. Fig. 1). If $p(r)$ is the loading on the plane $z = h$, the presence of body forces implies the continuity of displacements $u_r, u_z$ and shear stress $\sigma_{rz}$ and the discontinuity of normal stress $\sigma_{zz}$ following the condition:

$$\lim_{\epsilon \to 0^+} \left[ \sigma_{zz}(r, h - \epsilon) - \sigma_{zz}(r, h + \epsilon) \right] = p(r), \quad (1)$$

where $p$ is an arbitrary function so defined for $r \geq 0$ that the Hankel transform of order zero $p^H$ exists.

Over the plane $z = 0$, the stresses and displacements are continuous in the exterior of the crack ($r > r_0$); the boundary of which is stress-free, i.e. the stresses $\sigma_{zz}$ and $\sigma_{rz}$ take prescribed values over the surface of the crack, so that:

$$\sigma_{zz}(r, 0) = 0, \quad 0 \leq r < r_0, \quad (2)$$

The remaining boundary conditions assume that
the components of stress and of displacement vanish as \((r^2 + z^2)^{1/2} \to \infty\).

2. Resolution. — In order to solve this problem, we have to divide the space in three parts \(\Omega_i\):

\[
\Omega_1 = \{ (r, z) : r \in \mathbb{R}_+, h < z < +\infty \}, \\
\Omega_2 = \{ (r, z) : r \in \mathbb{R}_+, 0 < z < h \}, \\
\Omega_3 = \{ (r, z) : r \in \mathbb{R}_+, -\infty < z < 0 \}.
\]

Using the results given in a previous paper [4] and the notations of a precedent Letter [2], we know that there exists a potential function of the Love type \(\varphi\), defined by the components of stress tensor, such that in each part \(\Omega_i\), we have:

\[
\varphi(r, z) = \int_0^\infty \left[ A_i(m) e^{+s_1mz} + B_i(m) e^{+s_2mz} + C_i(m) e^{-s_1mz} + D_i(m) e^{-s_2mz} \right] J_0(mr) \, m \, dm,
\]

\[(r, z) \in \Omega_i, \quad i = 1, 2, 3.\]

If we take into account the boundary conditions on the plane \(z = h\), the condition at infinity and the continuity of the stresses \(\sigma_{zz}\) and \(\sigma_{rz}\) over the plane \(z = 0\), we have the following relations between the twelve functions \(A_i, B_i, C_i, D_i\):

\[
A_i(m) = B_i(m) = C_i(m) = D_i(m) = 0, \\
A_2(m) = e^{-\text{as}_1h} p(m)\left[2 m^3 \, ds_1(s_1^2 - s_2^2)\right], \\
B_2(m) = e^{-\text{as}_2h} p(m)\left[2 m^3 \, ds_2(s_2^2 - s_1^2)\right], \\
C_1(m) = e^{+\text{as}_1h} p(m)\left[2 m^3 \, ds_1(s_1^2 - s_2^2)\right] + C_2(m), \\
D_1(m) = e^{+\text{as}_2h} p(m)\left[2 m^3 \, ds_2(s_2^2 - s_1^2)\right] + D_2(m), \\
A_3(m) = A_2(m) + \frac{s_1 + s_2}{s_1 - s_2} C_2(m) + \frac{2 s_1}{s_1 - s_2} D_2(m), \\
B_3(m) = B_2(m) + \frac{s_2 + s_1}{s_2 - s_1} D_2(m) + \frac{2 s_2}{s_2 - s_1} D_2(m).
\]

We have to determine the functions \(C_2\) and \(D_2\). For this, by using the equations (2) and the continuity of the displacements in the exterior of the crack for \(z = 0\), we obtain a system of four integral equations:

\[
\int_0^\infty \left[ s_1 g_1 C_2(m) + s_2 g_2 D_2(m) - (g_1 e^{-\text{as}_1h} - g_2 e^{-\text{as}_2h}) \frac{p(m)}{2 m^3 \sqrt{d(s_1^2 - s_2^2)}} \right] m^4 J_0(mr) \, dm = 0, 0 < r < r_0, \\
\int_0^\infty \left[ s_2 p_1 C_2(m) + s_1 p_2 D_2(m) \right] m^3 J_0(mr) \, dm = 0, r_0 \leq r < \infty, \\
\int_0^\infty \left[ p_1 C_2(m) + p_2 D_2(m) + (p_1 s_2 e^{-\text{as}_1h} - p_2 s_1 e^{-\text{as}_2h}) \frac{p(m)}{2 m^3 \sqrt{d(s_1^2 - s_2^2)}} \right] m^4 J_1(mr) \, dm = 0, 0 < r < r_0, \\
\int_0^\infty \left[ p_1 C_2(m) + p_2 D_2(m) \right] m^3 J_1(mr) \, dm = 0, r_0 \leq r < \infty.
\]

This system can be solved if we notice the last two equations are two dual integral equations of the unknown function \(p_1 C_2 + p_2 D_2\). Having determined this function, then the first two equations are dual integral equations of the function \(C_2\) or \(D_2\). Thus, the solution of the system (6) is given by:

\[
C_2(m) = [s_1 I(m) - s_1 J(m) - K(m)] m^3 p_1(s_1^2 - s_2^2), \\
D_2(m) = [s_2 I(m) - s_2 J(m) - K(m)] m^3 p_2(s_2^2 - s_1^2),
\]

where the functions \(I, J, K\) depend on the loading and the geometrical parameters such that:

\[
I(m) = \frac{1}{\pi \sqrt{d(s_1 - s_2)}} \int_0^\infty \frac{\sin (mr_0)}{m r_0} \left( s_2 p_1 e^{-\text{as}_1h} - s_1 p_2 e^{-\text{as}_2h} \right) p(m) \sin (ar_0) \frac{dx}{a} \cos (mt) \, dt, \\
J(m) = \frac{1}{\pi \sqrt{d(s_1 - s_2)}} \int_0^\infty \left[ \int_0^\infty \left( s_2 p_1 e^{-\text{as}_1h} - s_1 p_2 e^{-\text{as}_2h} \right) p(m) \cos (at) \, dx\right] \cos (mt) \, dt.
\]
3. Stress intensity factors. — For further discussions interesting the Fracture Mechanics, we can calculate the different stress intensity factors defined by the limits :

\[
\begin{align*}
    k_1 &= \lim_{r \to r_0} [2 \pi (r - r_0)]^{1/2} \sigma_x(r, 0), \\
    k_2 &= \lim_{r \to r_0} [2 \pi (r - r_0)]^{1/2} \sigma_z(r, 0), \\
    k_3 &= \lim_{r \to r_0} [2 \pi (r - r_0)]^{1/2} \sigma_\theta(r, 0) = 0.
\end{align*}
\]

From these definitions and relations (5) and (7), we deduce :

\[
\begin{align*}
    k_1 &= -\frac{1}{\sqrt{\pi r_0}} \int_0^\infty \frac{d(s_1^2 - s_2^2)}{s_1} \left( g_1 e^{-as_1h} - g_2 e^{-as_2h} \right) p^H(m) \sin (mr_0) \, dm, \\
    k_2 &= \frac{1}{\sqrt{\pi r_0}} \int_0^\infty \frac{d(s_1^2 - s_2^2)}{s_2} \left( s_1 p_1 e^{-as_1h} - s_1 p_2 e^{-as_2h} \right) \left[ \frac{\sin (mr_0)}{mr_0} - \cos (mr_0) \right] p^H(m) \, dm.
\end{align*}
\]

Closed form solutions can be obtained for usual loadings of the cracked solid (uniform loads over a disc, point concentrated force...). For a point force \( p^H = P/2 \pi \), the expressions obtained through a limit operation agree with the results of the only existing isotropic solution given by Kassir and Sih [5].

Moreover, this solution makes possible to calculate the stress intensity factors in the case of two loads \( p_1 \) and \( p_2 \) applied on each side of the crack at distances \( h_1 \) and \( h_2 \). From the principle of superposition, we have the new factors by using for each \( k_i \) the expressions (10) :

\[
\begin{align*}
    k_1 &= k_1(p_1, h_1) + k_1(p_2, h_2), \\
    k_2 &= k_2(p_1, h_1) - k_2(p_2, h_2).
\end{align*}
\]

For equal and symmetrically spaced loads, we remark that the factor \( k_1 \) is double the value given by (10) and the factor \( k_2 \) is zero. These expressions agree with the results obtained in the Letter mentioned above [2].

References