Correlation functions of a 2D fully frustrated Ising system on a square lattice

M. Gabay

► To cite this version:

M. Gabay. Correlation functions of a 2D fully frustrated Ising system on a square lattice. Journal de Physique Lettres, 1980, 41 (18), pp.427-431. 10.1051/jphyslet:019800041018042700 . jpa-00231814

HAL Id: jpa-00231814

<https://hal.science/jpa-00231814>

Submitted on 1 Jan 1980

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LE JOURNAL DE PHYSIQUE-LETTRES

J. Physique — *LETTRES* 41 (1980) L-427 - L-431

15 SEPTEMBRE 1980, PAGE L-427

Classification
Physics Abstracts
75.10H

Correlation functions of a 2D fully frustrated Ising system on a square lattice

M. Gabay

Laboratoire de Physique des Solides (+), Université Paris-Sud, Centre d'Orsay, 91405 Orsay, France

(Reçu le 25 avril 1980, révisé le 25 juin, accepté le 23 juillet 1980)

Résumé. — On étudie les fonctions de corrélation à deux et quatre points d'un système de spins d'Ising sur un réseau carré généralisé ; la méthode suivie s'appuie sur l'expression conjecturée de l'aimantation spontanée de ce système. Pour certaines valeurs des constantes d'interaction, on obtient le modèle ZZD (un système complètement frustré) : sa fonction de corrélation à deux points dans la direction diagonale, $\omega_2(k, k)$, se comporte comme $\frac{1}{k^{1/2}} \cos \frac{\pi}{2} k$ rappelant par là le cas du réseau triangulaire antiferromagnétique de spins Ising.

Abstract. — We study two and four points correlation functions for two dimensional Ising spins on the generalized square lattice ; the method makes use of a conjectured expression for the spontaneous magnetization of this system. For special values of the interaction parameters, we obtain those of the so called ZZD model (a fully frustrated system) ; its two points correlation function in the diagonal direction, $\omega_2(k, k)$, behaves like $\frac{1}{k^{1/2}} \cos \frac{\pi}{2} k$ quite similarly to the case of the antiferromagnetic Ising triangular lattice already studied by Stephenson.

1. Introduction. — On a theoretical basis, many efforts have recently been made to understand the behaviour of spin glasses. A model can be constructed the hamiltonian of which reads :

$$H = - \sum_{\langle ij \rangle} J_{ij} S_i \cdot S_j$$

where S_i are spin variables at site i of a regular lattice (in the following we restrict ourselves to Ising spins on a 2D lattice) ; these interact *via* random bond variables J_{ij} taking on real values and the summation extends either over nearest neighbour or over all distinct pairs of sites. The troubles stem from the fact that the J_{ij} are quenched variables so that one has to average disorder out in a special way to extract the thermodynamic properties of the model. As a result, no convincing analytical solution has been given so far in the whole range of temperatures.

Various side attacks have therefore been attempted. The one we will be performing in the present note concerns a class of systems termed as « fully frustrated

systems » (FFS) [1, 2]. To study their properties, one still uses Hamiltonian (1) with nearest neighbour couplings. However the J_{ij} take on only two values $+1$ or -1 which are *periodically* distributed over the lattice. To that extent the model no longer represents an orthodox spin glass but retains one of its salient features, namely frustration. (FFS) would be a first approximation, « on the frustrated side » to true spin glasses.

Furthermore restoring translational invariance allows explicit analytical calculations. Indeed the case of the 2D antiferromagnetic Ising model on a triangular lattice is a well known example of (FFS) [3] ; (FFS) have also been studied on hypercubic [4] and D dimensional face centred cubic lattices [5]. In the following we investigate correlation functions of various 2D Ising (FFS). Our aim is to gain some insight into the zero temperature spin configurations in the presence of frustration and into the concept of universality. Besides, as we argue in the conclusion, our method can easily be generalized to the study of partially frustrated systems (see also ref. [6]) so that one can follow the behaviour of the system as one varies its content of frustration.

(+) Laboratoire associé au C.N.R.S.

2. Determination of 2 and 4 points correlation functions. — 2.1 TWO SPIN CORRELATIONS IN THE GENERAL CASE. — Let us consider the generalized

square Ising [7, 8] model pictured in figure 1. In the thermodynamic limit, its partition function per spin, z , is such that (see ref. [9], p. 214-221 ; ref. [10], p. 322) :

$$\ln \left(\frac{z}{2} \right) = \frac{1}{16 \pi^2} \int_0^{2\pi} \int_0^{2\pi} d\omega_1 d\omega_2 \ln \frac{1}{2} \{ C_1 C_2 C_3 C_4 + S_1 S_2 S_3 S_4 + 1 - S_2 S_4 \cos(\omega_1 + \omega_2) - S_1 S_3 \cos(\omega_1 - \omega_2) - (S_1 S_2 + S_3 S_4) \cos \omega_1 - (S_1 S_4 + S_2 S_3) \cos \omega_2 \} \quad (2)$$

Fig. 1. — The generalized square lattice with its four coupling constants J_1, J_2, J_3, J_4 . The arbitrary origin is called A ; B corresponds to the point of coordinates (k, k) . The symmetry direction is denoted by X .

with

$$C_i = \cosh 2 K_i, \quad S_i = \sinh 2 K_i, \quad K_i = J_i / k_B T.$$

The transition temperature is determined from

$$\sum_{i=1}^4 Gd(2 K_i) = \pi \quad \text{where} \quad Gd(x) = 2 \tan^{-1}(e^x) - \frac{\pi}{2}. \quad (3)$$

To evaluate alongside rows or diagonal correlations, one can alternatively use Pfaffian methods or relate the expression of the spontaneous magnetization back to the two-points correlation function.

In the present case, the latter procedure proves to be much simpler. We will briefly sketch the strategy and give some details of the calculations.

Let us first consider, as an example, the case of an anisotropic triangular lattice of N spins ; correlations along-side a row which we denote by X (see ref. [9], p. 125-137) may be obtained from the $N \times N$ Tœplitz determinant of elements $a_n^{(x)}$,

$$a_n^{(x)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega e^{-in\omega} (-e^{-i\omega}) \times \frac{(1 - A^{-1} e^{i\omega})(1 - B e^{i\omega})}{|(1 - A^{-1} e^{i\omega})(1 - B e^{i\omega})|} \quad (4)$$

A and B are functions of the K_i and of the direction X we are considering. Their exact expression may be found in references [9], p. 248-251 and [12]. The spontaneous magnetization J is then given by

$$J^8 = 1 - G^{-2} \quad \text{where} \quad G^{-2} = (A - B)^2 / (1 - AB)^2. \quad (5)$$

Reciprocally if we know the expression of J and are able to cast it under the form (5) we can work our way back to expression (4) and identify X . Let us then try and apply this idea to the case of the *rectangular* lattice (lattice with unequal horizontal and vertical couplings). We may easily work the program through and rediscover the fact that we would have obtained the $a_n^{(x)}$ by directly setting $K_{i=X=0}$ in the anisotropic triangular lattice. Also X is readily identified as a diagonal direction in the rectangular lattice. We know that the symmetries of that lattice would enable to compute an $N \times N$ Tœplitz determinant but, had we used Pfaffian techniques directly, we would have come out with at least a $2N \times 2N$ determinant with complicated elements and have had a hard time getting the correlation function from it.

We now turn to the case of the generalized square lattice. The expression of J has been conjectured by Syozi and Naya [13]. It reads :

$$J^8 = 1 - \left(\frac{\sum_{i=1}^4 S_i^2 + 2 \prod_{i=1}^4 C_i + 2 \prod_{i=1}^4 S_i + 2}{\sum_{i \neq j \neq k} S_i^2 S_j^2 S_k^2 + 2 \prod_{i=1}^4 S_i^2 + 2 \prod_{i=1}^4 C_i S_i + 2 \prod_{i=1}^4 S_i} \right). \quad (6)$$

Notice that the denominator of (6) may be rewritten as :

$$(S_1 S_4 C_2 C_3 + C_1 C_4 S_2 S_3)^2 - (S_1 S_4 - S_2 S_3)^2. \quad (7)$$

Thus

$$G^2 = \frac{(S_1 S_4 C_2 C_3 + C_1 C_4 S_2 S_3)^2 - (S_1 S_4 - S_2 S_3)^2}{\sum_{i=1}^4 S_i^2 + 2 \prod_{i=1}^4 C_i + 2 \prod_{i=1}^4 S_i + 2}. \quad (8)$$

Comparison with known results yields (Ref. [9], p. 139-140 and 249)

$$A = \left\{ \frac{S_1 S_4 C_2 C_3 + C_1 C_4 S_2 S_3}{\sqrt{\sum_{i=1}^4 S_i^2 + 2 \prod_{i=1}^4 C_i + 2 \prod_{i=1}^4 S_i + 2}} - G \right\} \frac{1}{\lambda - 1} \quad (9)$$

and

$$B = \frac{\lambda - 1}{\lambda + 1} A.$$

Using (5) and (8) gives

$$\lambda = \frac{C_1 C_4 + C_2 C_3}{\sqrt{\sum_{i=1}^4 S_i^2 + 2 \prod_{i=1}^4 C_i + 2 \prod_{i=1}^4 S_i + 2}}. \quad (10)$$

At that stage, the structure of (4) allows us to determine the expression of x , the coefficient associated with correlations in the X direction (see for instance Ref. [9] chap. 3, 3.4 for a definition of x). After some algebra we get

$$x = \frac{C_1 C_4 + C_2 C_3 - \sqrt{\sum_{i=1}^4 S_i^2 + 2 \prod_{i=1}^4 C_i + 2 \prod_{i=1}^4 S_i + 2}}{S_1 S_4 + S_2 S_3}. \quad (11)$$

It is easily checked that, by setting the appropriate K_i s equal to zero or infinity, one recovers all known results for regular 2D lattices.

It might, *a priori*, seem surprising that correlation function can be obtained from an $N \times N$ Tœplitz determinant, when the basic cell of the generalized square lattice consists of two blocks (which, by repeated translations, generate the whole lattice). But it is the underlying symmetry of the system which yields this property : under the interchange $K_1 \leftrightarrow K_4$ and $K_2 \leftrightarrow K_3$, the partition function z as well as A , B and x are left unchanged. Since it corresponds to making a symmetry with respect to a diagonal direction of the lattice (see Fig. 1), this is what we call a diagonal invariance.

Although one also has a row invariance (under the interchange $K_1 \leftrightarrow K_3$ and $K_2 \leftrightarrow K_4$) x as given by (11) defines unambiguously the diagonal direction since, when one sets all K_i s equal to the same value K , (11) yields $x = 0$ and (8), (9), (10) give

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega e^{in\omega} (-e^{-i\omega}) \frac{1 - S^2 e^{i\omega}}{|1 - S^2 e^{i\omega}|}$$

which are the relevant expressions for the case of diagonal correlations of the square lattice (Ref. [11], p. 195-196).

From the above discussion, we conclude that (8), (9), (10), (11) are the expressions pertaining to the case of diagonal correlations of the generalized lattice.

2.2 FULLY FRUSTRATED CASE. — We consider the situation where $K_2 = K_3 = K_4 = -K_1 = K$. This corresponds to the so called ZZD [14] model.

(3) reads $\tan^{-1}(e^{2K_c}) = \pi/2$ or $T_c = 0$ that is the transition point is pushed down to zero.

(2) becomes :

$$\ln\left(\frac{z}{2}\right) = \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\omega_1 d\omega_2 \ln \{ 1 + \sinh^2 \times \\ \times 2K(\cos^2(\omega_1 - \omega_2) + \sin^2(\omega_1 + \omega_2)) \}$$

and the zero point entropy is $S = 0.2916$.

We thus recover well known results.

Now : (8) gives $G^2 = \frac{-S_4}{C^2 + S^2}$ so that G is purely imaginary. We then choose to write :

$$G = \frac{-iS^2}{\sqrt{C^2 + S^2}}.$$

Since from (10) we have,

$$\lambda = \frac{C^2}{\sqrt{C^2 + S^2}}$$

we find

$$A = i \frac{S^2}{C^2 - \sqrt{C^2 + S^2}}$$

and

$$B = i \frac{S^2}{C^2 + \sqrt{C^2 + S^2}}.$$

In the limit $T = 0$, $A = B = i$ and (4) becomes

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{-\cos \omega}{|\cos \omega|} e^{-in\omega} d\omega = -\frac{2}{\pi n} \sin \frac{n\pi}{2}.$$

From reference [12] we deduce that the two points correlation function in the diagonal direction (k, k) is given by

$$\begin{aligned} \omega_2(k, k) &= \cos \frac{\pi k}{2} \left(\left(\frac{2}{\pi} \right)^{k/2} \prod_{s=1}^{k/2} \frac{\Gamma^2(s)}{\Gamma\left(s - \frac{1}{2}\right) \Gamma\left(s + \frac{1}{2}\right)} \right) \\ &\sim \frac{0.588}{k^{1/2}} \cos \frac{\pi k}{2} \left(1 - \frac{1}{8k^2} + \dots \right). \end{aligned} \quad (12)$$

When $T > 0$, one has to introduce a quantity θ such that :

$$\cos \theta = \frac{A^{-1} + B}{2\sqrt{A^{-1}B}};$$

now, in our case $\cos \theta = 0$ for $T > 0$ as well as for $T = 0$, so that

$$\omega_2(k, k) \sim \left(\frac{\pi}{2} \right)^{-1/2} \frac{1}{k^{1/2}} v^k \cos \frac{\pi k}{2}$$

where

$$v = \tanh K, \quad \text{for } T > 0.$$

2.3 FOUR POINTS CORRELATION FUNCTIONS. — We consider here correlations between pairs of spins at sites $(0, 0)$ $(1, 1)$ and pairs at sites (k, k) $(k + 1, k + 1)$. As was shown [15],

$$\omega_4(0, 1, k, k + 1) = a_0^2 - a_k a_{-k}$$

so that we have

$$\omega_4 = - \left(\frac{2}{\pi k} \sin \frac{k\pi}{2} \right)^2 \quad \text{for } T = 0$$

$$\omega_4 = - \frac{1}{2\pi} \frac{v^{2k}}{k^2} \sin^2 \frac{k\pi}{2} \quad \text{for } T > 0.$$

3. Analysis of the results and conclusion. — Just as in the case of the triangular lattice [16], $\omega_2(k, k)$ presents two characteristics :

— an oscillatory prefactor which reflects both frustration effects and the periodicity of the distribution of J_{ij} ; if one were to mimic a true spin glass by dividing the system into non interacting fully frustrated pieces, obtaining two points correlation functions would amount to average $\omega_2(k, k)$ in some way; this would give zero because of the cosine term. This qualitative result is just the property that one expects to hold in real spin glasses,

— without the oscillating term, we would have a critical point at zero temperature with an unexpected correlation function exponent $\eta = \frac{1}{2}$.

Besides, the similarity between results for the triangular antiferromagnetic and the ZZZ models suggests some universality for FFS in two dimensions. It is therefore reasonable to predict the following behaviour for the FFS on a honeycomb lattice at zero temperature :

$$\omega_2(k, k) \propto \frac{1}{k^{1/2}} \cos \frac{\pi}{3} k$$

$$\omega_4(k, k) = - \left(\frac{2}{\pi k} \sin \frac{k\pi}{3} \right)^2.$$

It would also be interesting to check the concept of universality for higher dimensions.

All the results we have quoted in the present paper may be used to obtain correlation functions for partially frustrated systems with the help of Utiyama's Chess board lattice [8, 9]. It is an extension of the generalized square lattice and the couplings of both systems can be related [9]. We may thus have an expression for the spontaneous magnetization in that case and then the correlation functions. (However we cannot solve by this method the problem of the Chess board frustrated model of Ref. [14].) Clearly what we have obtained in the present note and the possible extensions one could make rely on the correctness of the expression for the spontaneous magnetization [6]; the fact that it leads to sensible expressions for correlations in the case of regular lattices and particularly in the case of the square lattice gives it, in our opinion, further credence. Results for partially frustrated systems will be presented in a forthcoming paper.

Acknowledgments. — It is a pleasure to thank Drs. R. Bidaux, J. P. Carton and J. M. Maillard for valuable discussions and suggestions.

References

- [1] TOULOUSE, G., *Commun. Phys.* **2** (1977) 115.
  - [2] VILLAIN, J., *J. Phys. C* **10** (1977) 1717.
  - [3] WANNIER, G. H., *Phys. Rev.* **79** (1950) 357.
  - [4] DERRIDA, B., POMEAU, Y., TOULOUSE, G., VANNIMENUS, J.,
J. Physique **40** (1979) 617; *J. Physique* **41** (1980) 213.
  - [5] ALEXANDER, S., PINCUS, P., *J. Phys. A* **13** (1980) 263.
  - [6] Partially frustrated systems have been considered by
LONGA, L. and OLEŚ, A. M., *J. Phys. A* **13** (1980) 1031.
  - [7] YAMAMOTO, T. T., *Prog. Theor. Phys.* **6** (1951) 533.
  - [8] UTIYAMA, T., *Prog. Theor. Phys.* **6** (1951) 907; see also references [9] and [10].
  - [9] GREEN, H. S., HURST, C. A., *Order disorder phenomena* (Intersciences publishers).
  - [10] DOMB, C., GREEN, M. S., *Phase transitions and critical phenomena*, Vol. **1** (Academic Press).
  - [11] MCCOY, B. M., WU, T. T., *The two dimensional Ising model* (Harvard University Press).
  - [12] STEPHENSON, J., *J. Math. Phys.* **11** (1970) 420.
  - [13] SYOZI, I., NAYA, S., *Prog. Theor. Phys.* **24** (1960) 829.
  - [14] ANDRÉ, G., BIDAUX, R., CARTON, J. P., CONTE, R., DE SÈZE, L.,
J. Physique **40** (1979) 479; *J. Appl. Phys.* **50** (1979) 7345.
  - [15] STEPHENSON, J., *J. Math. Phys.* **7** (1966) 1123.
  - [16] STEPHENSON, J., *J. Math. Phys.* **11** (1970) 413.
-