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Critical transition to stochasticity for some dynamical systems

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Résumé. — Nous définissons un paramètre stochastique local pour des systèmes dynamiques simples et l'analysons avec les méthodes du groupe de renormalisation.

Abstract. — We define a local stochastic parameter for some simple dynamical systems and analyse it in the frame of renormalization group methods.

Recent experiments on the convection in a fluid layer heated from below [1] (Rayleigh-Bénard problem) suggest that subharmonic bifurcations might play an important role in the mechanism of the transition to turbulence. Sequences of subharmonic bifurcations leading to chaos are also numerically observed when studying differential equations [2] as e.g. Lorenz system. This motivates to have a good understanding of such bifurcations, even in the simplest models where they can be studied, namely iterations of maps of an interval.

We shall consider here one parameter families $\{f_R\}_R$ of smooth maps verifying the following properties :

1) For each R , $f_R([0, 1]) \subset [0, 1]$, $f(0) = f(1) = 0$ and f_R admits a unique critical point $\bar{x} \in]0, 1[$.

2) f_R depends continuously on R and R ranges in some interval $[R_{\min}, R_{\max}]$ with $f_{R_{\min}} \equiv 0$, $f_{R_{\max}}(\bar{x}) = 1$, in such a way that no periodic orbit disappears (this means that there is no simplification of the global dynamics when R increases). This condition is satisfied by families of piecewise differentiable maps and conjectured for families of smooth maps as e.g. $f_R(x) = Rx(1-x)$ [3].

These last years, many studies, both numerical and theoretical, have been devoted to these maps whose properties are now almost well understood. It is, in particular, well established that a non countable infinity of asymptotic behaviours, both periodic and aperiodic, exists for such maps. In fact, this is a *too fine* analysis from the experimental (real or numerical) point of view. Numerical experiments have suggested

a coarse analysis of the possible behaviours associated to this problem. It allows to introduce a new critical exponent (in the sense of [4]) describing the transition to chaos.

First, let us define a distinguished subset of $[0, 1]$ hereafter referred to as trapping region. Some more technical properties of this subset, together with related results, will be mentioned at end. We start from the elementary fact that $f_R([f_R^2(\bar{x}), f_R(\bar{x})])$ is included in $[f_R^2(\bar{x}), f_R(\bar{x})]$. We construct a subset of $[f_R^2(\bar{x}), f_R(\bar{x})]$, depending on R by steps, and displaying the same trapping property.

As we are interested in the onset of chaos, we suppose that $R > R_c$, where R_c is the value of R where the first asymptotically aperiodic behaviours appears. R_c corresponds also to the accumulation point of the first sequence of doubling bifurcations [4].

Then, we write $E_1(R) = [f_R^2(\bar{x}), f_R(\bar{x})]$ and, if $f_R^4(\bar{x}) < f_R^3(\bar{x})$, i.e. if f_R does not admit any cycle with odd period greater than one,

$$E_2(R) = [f_R^2(\bar{x}), f_R^4(\bar{x})] \cup [f_R^3(\bar{x}), f_R(\bar{x})].$$

More generally, if $E_{2^n}(R)$ is defined and if, for each of its 2^{n-1} connex components $[f_R^k(\bar{x}), f_R^l(\bar{x})]$, we have

$$f_R^k(\bar{x}) < f_R^{k+2^n}(\bar{x}) < f_R^{l+2^n}(\bar{x}) < f_R^l(\bar{x}), \quad (1)$$

we construct $E_{2^{n+1}}(R)$ by replacing each component of $E_{2^n}(R)$ by

$$[f_R^k(\bar{x}), f_R^{k+2^n}(\bar{x})] \cup [f_R^{l+2^n}(\bar{x}), f_R^l(\bar{x})] \quad (2)$$

thus obtaining a set with 2^n connex components $I_j(R)$, $j = 1, 2, \dots, 2^n$.

Except for pathological cases easily eliminated, when f_R admits cycles with period 2^{n-1} . Q and no

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cycle with period $2^{n-2} \cdot P$, Q and P odd and greater than one, $T(R) = E_{2^n}(R)$ is a trapping region in the sense that almost all orbit are eventually trapped by it. We denote by $\bar{R}_n$ the supremum of the values of R such that $E_{2^{n+1}}(R)$ exists. The sequence $\{\bar{R}_n\}_n$ is decreasing and converges to R_c . The variation of shape and size of this trapping region $T(R)$ when R is increased from R_c to $R_{\max}$ is schematically represented on figure 1. As a result of the renormalization group technics recently introduced in the study of maps of an interval [4], the number $N(R)$ of connex components of the trapping region $T(R)$ as a function of R behaves as

$$N(\bar{R}_n) \propto (\bar{R}_n - R_c)^{-\nu} \quad (3)$$

Fig. 1. — Rough construction of the stochastic region S_R as a function of the parameter R . The $\bar{R}_n$'s correspond to qualitative changes of S_R . The shadowed parts correspond to the complementary of S_R in $[0, 1]$. We have indicated expressions of the bounds of S_R : $f^n(\bar{x})$ as a function of R is represented as long as it bounds S_R .

where

$$\nu = \frac{\log 2}{\log \lambda}, \quad (4)$$

and λ is the eigenvalue greater than one of the renormalization group generator linearized at its fixed point, like in Wilson's theory. Let us recall that there is a Cantor set C invariant under f_{R_c} . On the other hand, R_c is the unique value of R such that $E_{2^n}(R)$ exists for each $n \geq 0$. In fact, we have

$$C = \bigcap_{n=0}^{\infty} (E_{2^n}(R_c)), \quad (5)$$

but, since for R close enough from R_c , $T(R)$ is slightly equal to some $E_{2^n}(R_c)$, we can also write

$$C = \lim_{R \rightarrow R_c} T(R) \quad (6)$$

so that, if we define the global scaling factor of C as

$$\rho = \lim_{n \rightarrow \infty} \frac{m(E_{2^{n+1}}(R_c))}{m(E_{2^n}(R_c))} \quad (7)$$

m being Lebesgue measure (total length here), we have also :

$$\rho = \lim_{n \rightarrow \infty} \frac{m(T(\bar{R}_{n-1}))}{m(T(\bar{R}_n))}, \quad (8)$$

and, using (3) and (4)

$$m(T(\bar{R}_n)) \propto (\bar{R}_n - R_c)^{\log \rho / \log \lambda}. \quad (9)$$

The universality of ρ [it depends only on the order of the zero of $f'_R(\bar{x})$] has been illustrated in table I.

Up to now, there is no analytic way to relate ρ to the scaling factor α of the renormalization group as defined in [4]. One can only construct various approximations up to any finite order.

Table I. — Numerical values of the first approximants of ρ for various functions. Here $\rho(n) = m(S_{R_c}^{n-1})/m(S_{R_c}^n)$.

z	1	1	1	3
f_R	$RX(1 - X)$	$R \sin \pi X$	$RX(1 - X^2)$	$R \sin \left(\frac{\pi}{2} \sin \pi X \right)$
R_c	3.569 945 671 83	0.865 579 268 94	2.302 283 462 71	0.946 765 896 11
$\rho(1)$	1.782 5	1.751 0	1.704 2	1.375 3
$\rho(2)$	1.732 8	1.739 1	1.750 7	1.358 7
$\rho(3)$	1.743 4	1.741 9	1.739 1	1.362 4
$\rho(4)$	1.740 6	1.741 3	1.741 9	1.359 8
$\rho(5)$	1.741 3	1.741 3	1.741 3	1.360 7
$\rho(6)$	1.741 3	1.741 3	1.741 3	1.360 2
$\rho(7)$	1.741 3	1.741 3	1.741 3	1.360 5
$\rho(8)$	1.741 3	1.741 3	1.741 3	1.360 5

Now, let us recall that what is observed (with a computer) when $R \gtrsim R_c$ may be interpreted as some *noisy 2^n periodic motion*, except for a number of periodic behaviours which is finite for any given numerical precision [5]. Thus, (9) gives precisely the critical behaviour of this apparent noise, with the restriction (a general rule, but for different reasons in critical phenomena) that the validity of (9) is bounded away from R_c by an arbitrarily small but finite interval in R (due to finite numerical precision).

The number of connex components of $T(R)$ has an other meaning with respect to the analysis of chaos. Complete chaos is defined by the fact that all autocorrelation functions :

$$F_g(n) = \langle g(x) \cdot g(f^n(x)) \rangle - \langle g^2(x) \rangle, \quad (10)$$

with :

$$\langle A(x) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N A(f^i(x)), \quad (11)$$

and : x out of a set of measure zero (12)

asymptotically vanish. This can occur only if f_R admits cycles with odd period greater than one, i.e.

$$T(R) = E_1(R).$$

If more generally $T(R) = E_{2^n}(R)$, the 2^{n-1} intervals $I_j(R)$ are invariant under $f_R^{2^{n-1}}$ so that autocorrelation can vanish only if calculated with $f_R^{2^{n-1}}$ instead of f_R . When they are computed with f_R , we get *oscillations with coarse period 2^{n-1}* [6]. Let us end with some remarks.

1) It is usual to characterize global chaos in (non conservative) dynamical systems by the positive character of topological entropy [7] which, for maps of the kind we consider, may be computed as [8]

$$h(f_R) = \lim_{n \rightarrow \infty} \frac{1}{n} \log C_n(R), \quad (13)$$

where C_n is the number of inverse images of $\bar{x}$ under f_R^n . We can construct a local quantity :

$$h_R(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N_R(x, n), \quad (14)$$

where $N_R(x, n)$ is the number of inverse images of x under f_R^n . Then it can be shown that [9]

$$\begin{aligned} h_R(x) &= 0 & \text{if } x \notin T(R) \\ h_R(x) &= h(f_R) & \text{if } x \in T(R) \end{aligned} \quad (15)$$

except for some pathological cases eliminated, e.g. if f_R is C^3 with negative Schwarzian derivative [10].

Thus, $T(R)$ appears as the support of a local topological entropy or as the support of the horseshoe effect [11] which was demonstrated to exist for continuous map of an interval in [12] (see also [8]).

2) With some additional conditions on f_R (e.g. $f_R \in C^3$ with negative Schwarzian derivative), $T(R)$ reveals to be the unstable manifold (in the sense of [13]) of the first created cycle with period 2^n if $T(R) = E_{2^n}$. Then, it seems worth recalling that the strange attractor in Hénon $\mathbb{R}^2$ mapping [14]

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}, \quad (16)$$

if it exists, is the adherence of the unstable manifold of some cycle [15]. F. Marotto recently showed how powerful it may be to consider such mapping as a perturbation of a map from $\mathbb{R}$ to $\mathbb{R}$ [16]. He used this idea to show the existence of homoclinic points for (16) when b is small enough. Let us note that $\bar{R}_n$ is the one dimensional equivalent of a such that for (16), a first tangent homoclinic point appears in the invariant manifolds of some cycle with period 2^n . These values, say $\bar{a}_n(b)$, separate values of a where the apparent Hénon strange attractor has 2^n connex components, for different values of n [17]. This is to be compared to what we called *noisy 2^n periodic motion* in one dimension. However, these $\bar{a}_n(b)$ for $b \neq 0$ cannot generally be used to separate parameter domains according to the existence of cycles with period of the form $2^n \cdot P$, at least for b large enough.

3) Various numerical simulations performed on differential equations exhibiting stochastic behaviour [18] seem to be qualitatively interpretable in terms of the coarse analysis presented here. In fact, this is not a surprise in view of the second remark and the evident connexion between flows and diffeomorphisms *via* a first return map.

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