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A sufficient condition for the existence of bound states in a potential without spherical symmetry (*)

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Résumé. — Dans cette note, nous donnons une condition suffisante pour qu'un potentiel purement attractif sans symétrie sphérique ait au moins un état lié.

Abstract. — In this note we give a sufficient condition for a purely attractive potential without spherical symmetry to have at least one bound state.

In principle, it is not difficult to find sufficient conditions for the existence of bound states of a particle in a potential. If the wave function satisfies the tridimensional Schrödinger equation

$$H\psi = (-\Delta + V(\mathbf{X}))\psi = E\psi \quad (1)$$

with $V(\mathbf{X}) \rightarrow 0$ as $|\mathbf{X}| \rightarrow \infty$, it would be sufficient to find a trial wave function $\Phi(\mathbf{X})$ such that $(\Phi, H\Phi)$ is negative. However, it would be preferable to obtain conditions in terms of simple integrals of the potential. For a spherically symmetric and purely attractive potential, Calogero [1] obtained, some time ago, the sufficient condition

$$R^{-1} \int_0^R |V(r)| r^2 dr + R \int_R^\infty |V(r)| dr \geq 1 \quad (2)$$

in which the choice of R is arbitrary. In this formula the constants are optimal as one can see on the example $-V_0 \delta(r - r_0)$ by choosing R close to r_0 .

We give in this letter the generalization of (2) to the non spherically symmetric case, the first step toward the generalization to the N -body case, assuming that V is everywhere attractive. One of us (K.C. [2]) has already obtained sufficient conditions for the

non spherical case, for example

$$\text{tr } K^{(3)} - \text{tr } K^{(2)} \geq 0 \quad (3)$$

where

$$K(\mathbf{X}, \mathbf{X}') = |V(\mathbf{X})|^{1/2} \frac{1}{|\mathbf{X} - \mathbf{X}'|} |V(\mathbf{X}')|^{1/2} \quad (4)$$

which, unfortunately, necessitates the calculation of an 9-uple integral, and is also not optimal.

We start here from the remark that, if there are no bound states in a potential which decreases fast enough at infinity and if this potential is attractive, the solution of the Schrödinger equation at zero energy

$$\psi(\mathbf{X}) = 1 - (4\pi)^{-1} \int \frac{V(\mathbf{Y})}{|\mathbf{X} - \mathbf{Y}|} \psi(\mathbf{Y}) d^3Y \quad (5)$$

is necessarily positive; indeed, the Born series gives the solution of (5), and all its terms are positive (see also [3]). Therefore, if we can show that, under certain conditions on the potential, the equation (5) does not admit a strictly positive solution, we have proven the existence of bound states. For this purpose, we consider the sphere $|\mathbf{X}| = R$, and divide the space in two regions. Let

$$M_1 = \inf_{|\mathbf{X}| \leq R} |\psi(\mathbf{X})|, \quad M_2 = \inf_{|\mathbf{X}| \geq R} |\psi(\mathbf{X})|. \quad (6)$$

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It is now easy to obtain from (5), by remembering that the potential is everywhere attractive,

$$M_1 \geq 1 + M_1 \inf_{|X| \leq R} \int_{|Y| \leq R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|X - Y|} + M_2 \inf_{|X| \leq R} \int_{|Y| > R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|X - Y| |Y|} \quad (7)$$

and

$$M_2 \geq R + M_1 \inf_{|X| \geq R} |X| \int_{|Y| \leq R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|X - Y|} + M_2 \inf_{|X| \geq R} |X| \int_{|Y| > R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|X - Y| |Y|}. \quad (8)$$

To go further, we must use the two following lemmas :

Lemma 1. — If $W(Y) \geq 0$, and

$$f(r) = \inf_{|X|=r} F(X), \quad F(X) = \int \frac{d^3 Y}{4\pi} \frac{W(Y)}{|X - Y|}, \quad (9)$$

$f(r)$ is a decreasing function of r .

Lemma 2. — $f(r)$ defined by (9) is such that $r f(r)$ is an increasing function of r .

To prove lemma 1, it suffices to notice that $F(X)$ defined by (9) is a superharmonic function [4] simply because $\Delta F = -W < 0$. Therefore, the same is true for $f(r)$ defined as the infimum of such functions. This proves that $f(r)$ is a decreasing function.

For the proof of lemma 2, we first assume that W is sufficiently regular (double differentiability). It follows then that $F(X)$ and $f(r)$ are also regular, and the superharmonicity of $f(r)$ implies

$$r \Delta f = r^{-1} \frac{d}{dr} \left(r^2 \frac{d}{dr} f \right) = (2 r f' + f'') \leq 0. \quad (10)$$

Integrating this equation from r to infinity, we obtain

$$(r f' + f) > \lim_{r \rightarrow \infty} (r f' + f). \quad (11)$$

Now, it is easy to see that if W is sufficiently decreasing at infinity, the r.h.s. of (11) vanishes. This proves the lemma 2. In the general case where W is only an L^2 function, one reduces the same conclusion by approximating it by C^2 functions.

Starting from the above lemmas, we deduce that the infimums of the integrals in (7) and (8) are reached for $|X| = R$. If we denote these infimums by J_1 and J_2 , (7) and (8) are written simply

$$M_1 > 1 + M_1 J_1 + M_2 J_2$$

and

$$M_2 > R(1 + M_1 J_1 + M_2 J_2).$$

Therefore, a necessary condition for the absence of bound states would be $J_1 + R J_2 < 1$. It follows that a sufficient condition for having a bound state will be

$$\inf_{|X|=R} \int_{|Y| \leq R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|X - Y|} + R \inf_{|X|=R} \int_{|Y| > R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|X - Y| |Y|} \geq 1. \quad (12)$$

In the case of a spherically symmetric potential, this gives back the condition (2) of Calogero. We see therefore that (12) is also optimal by considering again the Dirac δ -function potential. Also, one can simplify somewhat the calculations and replace (12) by a stronger condition by noticing that, for $|X| = R$, we have $|X - Y| < R + |Y|$. One obtains then the sufficient condition

$$\int_{|Y| \leq R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{|Y| + R} + R \int_{|Y| > R} \frac{d^3 Y}{4\pi} \frac{|V(Y)|}{(R + |Y|) |Y|} \geq 1. \quad (13)$$

The above conditions can easily be generalized for the case of $m > 3$ dimensions by using the appropriate Green's function [4]. One obtains then the sufficient condition

$$\inf_{|X|=R} \int_{|Y| \leq R} \frac{d^m Y}{C_m} \frac{|V(Y)|}{|X - Y|^{m-2}} + R^{m-2} \inf_{|X|=R} \int_{|Y| > R} \frac{d^m Y}{C_m} \frac{|V(Y)|}{|X - Y|^{m-2} |Y|^{m-2}} \geq 1 \quad (14)$$

where $C_m = (4\pi^{m/2})/\Gamma(m/2 - 1)$.

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