A sufficient condition for the existence of bound states in a potential without spherical symmetry

K. Chadan, A. Martin

To cite this version:


HAL Id: jpa-00231760
https://hal.archives-ouvertes.fr/jpa-00231760
Submitted on 1 Jan 1980
A sufficient condition for the existence of bound states in a potential without spherical symmetry (*)

K. Chadan and A. Martin (**)

Laboratoire de Physique Théorique et Hautes Energies (***)
Université de Paris Sud, 91405 Orsay, France

(Reçu le 23 janvier 1980, accepté le 6 mars 1980)

Résumé. — Dans cette note, nous donnons une condition suffisante pour qu'un potentiel purement attractif sans symétrie sphérique ait au moins un état lié.

Abstract. — In this note we give a sufficient condition for a purely attractive potential without spherical symmetry to have at least one bound state.

In principle, it is not difficult to find sufficient conditions for the existence of bound states of a particle in a potential. If the wave function satisfies the tridimensional Schrödinger equation

\[ H\psi = (-\Delta + V(X))\psi = E\psi \] (1)

with \( V(X) \to 0 \) as \( |X| \to \infty \), it would be sufficient to find a trial wave function \( \Phi(X) \) such that \( (\Phi, H\Phi) \) is negative. However, it would be preferable to obtain conditions in terms of simple integrals of the potential. For a spherically symmetric and purely attractive potential, Calogero [1] obtained, some time ago, the sufficient condition

\[ R^{-1} \int_0^R |V(r)| r^2 dr + R \int_R^\infty |V(r)| dr \geq 1 \] (2)

in which the choice of \( R \) is arbitrary. In this formula the constants are optimal as one can see on the example \(-V_0 \delta(r - r_0)\) by choosing \( R \) close to \( r_0 \).

We give in this letter the generalization of (2) to the non spherically symmetric case, the first step toward the generalization to the \( N \)-body case, assuming that \( V \) is everywhere attractive. One of us (K.C. [2]) has already obtained sufficient conditions for the non spherical case, for example

\[ \text{tr} K^{(3)} - \text{tr} K^{(2)} \geq 0 \] (3)

where

\[ K(X, X') = |V(X)|^{1/2} \frac{1}{|X - X'|} |V(X')|^{1/2} \] (4)

which, unfortunately, necessitates the calculation of an \( 9 \)-uple integral, and is also not optimal.

We start here from the remark that, if there are no bound states in a potential which decreases fast enough at infinity and if this potential is attractive, the solution of the Schrödinger equation at zero energy

\[ \psi(X) = 1 - (4\pi)^{-1} \int \frac{V(Y)}{|X - Y|} \psi(Y) d^3Y \] (5)

is necessarily positive; indeed, the Born series gives the solution of (5), and all its terms are positive (see also [3]). Therefore, if we can show that, under certain conditions on the potential, the equation (5) does not admit a strictly positive solution, we have proven the existence of bound states. For this purpose, we consider the sphere \( |X| = R \), and divide the space in two regions. Let

\[ M_1 = \inf_{|X| \leq R} |\psi(X)|, \quad M_2 = \inf_{|X| > R} |X \psi(X)|. \] (6)

(*) La version française de cet article a été acceptée aux Comptes Rendus de l'Académie des Sciences et est insérée dans le n° du 18 février 1980.

(**) Permanent address: Theory Division, CERN, 1211 Genève 23, Suisse.

(***) Laboratoire Associé au C.N.R.S.
It is now easy to obtain from (5), by remembering that the potential is everywhere attractive,

\[ M_1 \geq 1 + M_1 \inf_{|X| \leq R} \int_{|Y| \leq R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|X - Y|} + M_2 \inf_{|X| \leq R} \int_{|Y| > R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|X - Y| |Y|} \]

and

\[ M_2 \geq R + M_1 \inf_{|X| \geq R} \int_{|Y| \leq R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|X - Y|} + M_2 \inf_{|X| \geq R} \int_{|Y| > R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|X - Y| |Y|} \]

To go further, we must use the two following lemmas:

**Lemma 1.** If \( W(Y) \geq 0 \), and

\[ f(r) = \inf_{|X| = r} F(X), \quad F(X) = \int \frac{d^3Y}{4\pi} \frac{W(Y)}{|X - Y|}, \quad \text{(9)} \]

\( f(r) \) is a decreasing function of \( r \).

**Lemma 2.** \( f(r) \) defined by (9) is such that \( r f(r) \) is an increasing function of \( r \).

To prove lemma 1, it suffices to notice that \( F(X) \) defined by (9) is a superharmonic function [4] simply because \( \Delta F = -W < 0 \). Therefore, the same is true for \( f(r) \) defined as the infimum of such functions. This proves that \( f(r) \) is a decreasing function.

For the proof of lemma 2, we first assume that \( W \) is sufficiently regular (double differentiability). It follows then that \( F(X) \) and \( f(r) \) are also regular, and the superharmonicity of \( f(r) \) implies

\[ r \Delta f = r^{-1} \frac{d}{dr} \left( r^2 \frac{d}{dr} f \right) = (2 r f'' + f') \leq 0. \]

Integrating this equation from \( r \) to infinity, we obtain

\[ (r f' + f) \bigg|_{r=\infty} = \lim_{r \to \infty} (r f' + f) \quad \text{(11)} \]

Now, it is easy to see that if \( W \) is sufficiently decreasing at infinity, the r.h.s. of (11) vanishes. This proves the lemma 2. In the general case where \( W \) is only an \( L^2 \) function, one reduces the same conclusion by approximating it by \( C^2 \) functions.

Starting from the above lemmas, we deduce that the infimums of the integrals in (7) and (8) are reached for \( |X| = R \). If we denote these infimums by \( J_1 \) and \( J_2 \), (7) and (8) are written simply

\[ M_1 > 1 + M_1 J_1 + M_2 J_2 \]

and

\[ M_2 > R(1 + M_1 J_1 + M_2 J_2) \]

Therefore, a necessary condition for the absence of bound states would be \( J_1 + R J_2 < 1 \). It follows that a sufficient condition for having a bound state will be

\[ \inf_{|X| = R} \int_{|Y| \leq R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|X - Y|} + R \inf_{|X| \geq R} \int_{|Y| > R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|X - Y| |Y|} \geq 1. \]

In the case of a spherically symmetric potential, this gives back the condition (2) of Calogero. We see therefore that (12) is also optimal by considering again the Dirac \( \delta \)-function potential. Also, one can simplify somewhat the calculations and replace (12) by a stronger condition by noticing that, for \( |X| = R \), we have \( |X - Y| \leq R + |Y| \). One obtains then the sufficient condition

\[ \int_{|Y| \leq R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{|Y| + R} + R \int_{|Y| > R} \frac{d^3Y}{4\pi} \frac{|V(Y)|}{R + |Y| |Y|} \geq 1. \]

The above conditions can easily be generalized for the case of \( m > 3 \) dimensions by using the appropriate Green’s function [4]. One obtains then the sufficient condition

\[ \inf_{|X| = R} \int_{|Y| \leq R} \frac{d^m Y}{C_m} \frac{|V(Y)|}{|X - Y|^{m-2}} + R^{m-2} \inf_{|X| \geq R} \int_{|Y| > R} \frac{d^m Y}{C_m} \frac{|V(Y)|}{|X - Y|^{m-2} |Y|^{m-2}} \geq 1 \]

where \( C_m = (4\pi^{m/2}) / \Gamma(m/2 - 1) \).

**Acknowledgments.** - We would like to thank P. C. Sabatier for a useful remark and discussions.

**References**