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ORBITAL HYDRODYNAMICS OF THE ANISOTROPIC A PHASE OF ^3He FOR LARGE ANGLES OF ROTATION OF THE ORBITAL MOMENTUM

J. M. DELRIEU

S.P.S.R.M., C.E.N. Saclay, BP 2, 91190 Gif-sur-Yvette, France

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Résumé. — On présente une théorie de l'hydrodynamique linéaire décrivant les rotations du paramètre d'ordre et les courants de masses dans la phase A de l'hélium trois superfluide. Elle ne présente aucune des ambiguïtés liées à l'utilisation de la phase φ . Ainsi cette théorie permet de décrire sur des distances macroscopiques toutes les rotations de l'axe orbital $\hat{\mathbf{l}}$, quel que soit leur angle, beaucoup plus simplement que la théorie localement équivalente de Graham.

Abstract. — A linear hydrodynamic theory of the orbital order parameter rotations and mass transport in the A phase of superfluid ^3He is proposed which is free of the problems related to the definition of the phase φ , and thus, to describe large angle rotations of the order parameter orbital symmetry axis $\hat{\mathbf{l}}$, is simpler and more adequate than the locally equivalent theory of Graham.

Starting from conservation laws and from the structure of the order parameter Graham [1] and Seiden [2] have derived hydrodynamic equations of the anisotropic A phase. They describe the orbital part of the order parameter of the A phase by the superposition of a superfluid phase φ canonically conjugated with the particle number and an axial unit vector $\hat{\mathbf{l}}$ (in the direction of the net orbital momentum of the triplet pairs). More exactly the orbital part of the order parameter (Fig. 1) is defined by two orthonormal unitvectors (or bivector) [3]. $\hat{\Delta}'$ and $\hat{\Delta}''$ with :

$$\hat{\Delta}'^2 = 1, \quad \hat{\Delta}''^2 = 1, \quad \hat{\Delta}' \cdot \hat{\Delta}'' = 0, \quad \hat{\mathbf{l}} = \hat{\Delta}' \wedge \hat{\Delta}'' . \quad (1)$$

The phase φ gives the orientation of $\hat{\Delta}'$ in the plane perpendicular to $\hat{\mathbf{l}}$, if we take some origin axis $\hat{\mathbf{x}}$ in this plane. Mermin and Ho [4] have pointed out that because the angle φ is not uniquely well defined we must be very careful in the minimization of the free energy of the superfluid liquid. In this paper we explain first why the linear hydrodynamics of Graham and Seiden are not convenient to describe large angle rotations of $\hat{\mathbf{l}}$ and secondly, we derive strictly equivalent linear hydrodynamic equations of the A phase which describe more easily on macroscopic scale large angle rotations.

We shall (as did Graham) consider linear hydrodynamics only. In his note 11 Graham [1] points out

FIG. 1. — $\hat{\mathbf{l}}$ is perpendicular to the surface of a sphere as an example. We generate the different origin axis $\hat{\mathbf{x}}(r_i)$ along the two paths r_1, r_2 to r_3 or r_1 to r_3 by infinitesimal local parallel translations. At the point r_3 the axis $\hat{\mathbf{x}}_{123}$ of path r_1, r_2 to r_3 is in this example perpendicular to the axis $\hat{\mathbf{x}}_{13}$ of path r_1 to r_3 (because we choose one eighth of a sphere). Thus we cannot define rigorously the origin axis $\hat{\mathbf{x}}(r_i)$ for measuring the phase φ of the order parameter. The function φ is geometrically not defined when $\hat{\mathbf{l}}$ is not uniform.

that the rotation angles φ of the orbital part of the bivector order parameter are not unique, since finite rotations are, in general, non commuting operations. However, since we are here only concerned with linear hydrodynamics, we need to consider infinitesimal rotations which do commute to first order and thus do not

give rise to ambiguities of this kind. For this geometrical reason, the linear hydrodynamic equations of Graham cannot be integrated as they stand for large rotations of $\hat{\mathbf{l}}$ over all the sample. They must be supplemented by very complex non linear equations in order to assure the existence of the order parameter [4, 15]. For an isotropic superfluid the equivalent compatibility equation $\text{rot}(\nabla\varphi) = 0$ or $\text{rot } v_s = 0$ is included in the linear hydrodynamics. These compatibility equations [4, 8] of the superfluid A phase of ^3He are very non linear and very complex; when we take as variables $\varphi, \hat{\mathbf{l}}$, we have to handle surprising equations such as $\text{rot} \{ \nabla\delta\varphi \} = f(\nabla\hat{\mathbf{l}}) \neq 0$. Thus, as in reference [4] we can ask if *the linear hydrodynamics of reference [1] are adequate even when flow velocities are small*. In this paper we give a positive answer: they are adequate; it is sufficient to take as variables the true order parameter in order to suppress all the compatibility equations and obtain linear hydrodynamic equations identical to first order to that of reference [1]. For practical applications, as for liquid crystals we have to integrate only these linear equations without additional compatibility equations.

Figure 1 shows the nature of the geometrical ambiguity in the definition of the phase φ . At point r_1 the phase φ_1 gives the orientation of the vector $\hat{\Delta}'$ (i.e. of the bivector $(\hat{\Delta}', \hat{\Delta}'')$) in the plane perpendicular to $\hat{\mathbf{l}}_1$. For this reason we need at this point a reference axis $\hat{\mathbf{x}}_1$ perpendicular to $\hat{\mathbf{l}}_1$; the angle φ is the angle between $\hat{\Delta}'$ and $\hat{\mathbf{x}}_1$. As used by Graham [1] (or first by De Gennes [3]), in an infinitesimal translation from r_1 to r_2 , a infinitesimal translation of $\mathbf{x}_1$ gives $\mathbf{x}_2$. In figure 1 we choose this definition of translation to generate the reference axis $\hat{\mathbf{x}}(\mathbf{r})$ on a closed path drawn on a sphere (or a more general curved surface) taken perpendicular to $\hat{\mathbf{l}}$. As shown the orientation is a function of the path from r_1 to r_2 (and of the curvature of the surface) all the time that the direction of $\hat{\mathbf{l}}$ is not uniform. Thus the phase φ is not defined, for finite rotation of $\hat{\mathbf{l}}$. This ambiguity does not change as we change the length over which the rotation occurs; its nature is purely topological. There are different mathematical formulations of this problem [4, 8].

Graham in the Gibbs relation (6) of his paper writes

$$\frac{h}{2m} \lambda^s d(\nabla\varphi) = d\varepsilon_0$$

as the contribution of a phase gradient to the free energy ε_0 . After that he uses the relation

$$\frac{d}{dt}(\nabla\varphi) = \nabla\left(\frac{d\varphi}{dt}\right) \quad (2)$$

to calculate $d\varepsilon_0/dt$ and after partial integration the hydrodynamic reversible currents. The relation (2) of commutativity of derivations holds only for a well defined function φ and is wrong for the ambiguous function φ . This comes from the definition: $d\varphi = \Delta'' \cdot d\Delta'$ which gives:

$$\dot{\varphi} = \Delta'' \cdot \dot{\Delta}' \quad \text{and} \quad \nabla\varphi = \Delta_j'' \nabla\Delta_j' \quad (3)$$

and taking the derivative of these last relations:

$$\begin{aligned} \nabla_i \dot{\varphi} &= \nabla_i \Delta_j'' \dot{\Delta}_j' + \Delta_j'' \nabla_i \dot{\Delta}_j', \\ \frac{d(\nabla_i \varphi)}{dt} &= \dot{\Delta}_j'' \nabla_i \Delta_j' + \Delta_j'' \nabla_i \dot{\Delta}_j' \neq \nabla_i \dot{\varphi}. \end{aligned} \quad (4)$$

For arbitrary movement of Δ', Δ'' they are different by second-order terms in the derivatives of the order parameter. In a linearized theory this second-order term does not contribute in the Gibbs relation. Thus as in reference [1], locally at each point we can use the phase $\delta\varphi$ which is defined only to first order. For practical applications, we need to integrate the linear equations of reference [1]. For this reason we need to know how to connect from point to point the different phases $\delta\varphi(\mathbf{r})$ which are only defined locally to first order. The solution of this problem is given by compatibility equations [4, 8] non linear in the derivatives of the order parameter of an unpleasant nature such as [4]: $\text{rot} \{ \nabla(\delta\varphi) \} = f(\nabla\hat{\mathbf{l}}) \neq 0$. For practical results we need to integrate on the same footing the linear equations of reference [1] and the very complex compatibility equations of references [4, 8] when $\hat{\mathbf{l}}$ wanders slowly through large angles. The simplicity of using linear hydrodynamic equations is practically suppressed. If we forget the compatibility equations and solve the equations of reference [1] as they stand with the wrong supposition that φ is a well defined phase, we obtain the hydrodynamic movement of a hypothetical anisotropic superfluid phase completely different from the real A phase of ^3He , but perfectly possible: the phase φ would be well defined and the rotations around the anisotropic axis $\hat{\mathbf{l}}$ would be completely different from a phase change. The properties of this phase would be completely different: for example Toulouse and Kleman [8] have shown that the structure of defects of the true A phase has nothing to do with the defects of this hypothetical phase. It is surprising that the same linear equations of reference [1] can describe two quite different superfluid phases simply by forgetting or adding to them the compatibility equations of references [4] and [8].

We show here that by adding adequate second order terms to the equations of reference [1] we can find linear equations in the derivatives of the order parameter which can be integrated as they stand without compatibility equations, because we simply use the true order parameter. We emphasize again that our equations are identical to that of reference [1] to first order. The practical difference is that the compatibility equations are included inside these linear equations, simplifying the calculations.

The basic physical principles which have been used at first for nematic liquid crystals [5] and for isotropic superfluid [6, 7] are the same as that used by Graham in reference [1].

We simply take the correct definition of the order parameter that is we consider as a complex director

$$\Delta = \hat{\Delta}' + i\hat{\Delta}'' \quad \text{with} \quad \hat{\mathbf{l}} = \frac{i}{2} \Delta \wedge \Delta^* . \quad (5)$$

The formulae (1) are equivalent to the two conditions :

$$\Delta^2 = 0 , \quad \Delta \cdot \Delta^* = 2 . \quad (6)$$

They are equivalent to the condition $\mathbf{n}^2 = 1$ in nematic liquid crystal [5]. On the other hand this complex vector is similar to the complex order parameter of an isotropic superfluid : the energy $\varepsilon(\nabla\Delta)$ is invariant with respect to a gauge transformation, i.e. a phase change :
if

$$\Delta_2(\mathbf{r}) = \left[\exp \left(i \frac{2m}{\hbar} \varphi(\mathbf{r}) \right) \right] \Delta_1(\mathbf{r}) \quad (7)$$

we have :

$$\varepsilon(\nabla\Delta(\mathbf{r})) = \varepsilon \left(\left(\nabla - \frac{2m}{\hbar} i \nabla\varphi \right) \Delta_1(\mathbf{r}) \right) . \quad (8)$$

We remark that Galilean invariance is a particular case of gauge invariance with $\nabla\varphi = \frac{2m}{\hbar} \mathbf{u}$ where $\mathbf{u}$ is the relative velocity of the frame (2) with respect to (1).

All the hydrodynamic equations must be invariant with respect to this gauge transformation. The difference with anisotropic superfluid [6, 7] is that we cannot define a unique phase φ for the three components of the complex vector Δ (except when $\mathbf{l}$ is uniform in space). Thus our derivation of the linear hydrodynamic equations is, if we forget that Δ is a vector, nearly identical to that of Khalatnikov [6, 7], simply replacing the phase φ by the superfluid wave function ψ with the condition $|\psi|^2 = \text{Const.}$ On the other hand if Δ is real, it is similar to a nematic.

Following Graham [1] we take the three fundamental conservation laws of mass density ρ , momentum density $\mathbf{g}$ (or current) and energy density ε and their appropriate currents :

$$\dot{\rho} + \nabla \cdot \mathbf{g} = 0 , \quad \dot{\mathbf{g}}_i + \nabla_j \sigma_{ij} = 0 , \quad \dot{\varepsilon} + \nabla_j j_{j\varepsilon} = 0 . \quad (9)$$

We seek the most general equation of evolution of Δ with time compatible with the conservation law (9) and relations (6), (7), (8). First, we write the corresponding Gibbs relation :

$$T d(\rho S) = d\varepsilon - \mu d\rho - V^n d\mathbf{g} - h_{ij}^* d(\nabla_j \Delta_i) - h_{ij} d(\nabla_j \Delta_i^*) . \quad (10)$$

This gives the production of entropy S for small change of ε , ρ , $\mathbf{g}$ and $\nabla_j \Delta_i$. h_{ij}^* and h_{ij} are the thermodynamically conjugated quantities of $\nabla_j \Delta_i$ and $\nabla_j \Delta_i^*$ respectively, which will be determined below.

The normal velocity V^n which is the canonical conjugate of $\mathbf{g}$ is defined by the Gibbs relation, i.e. $V^n = 0$ in the frame where the energy $\varepsilon = \varepsilon_0$ is a function of only ρ , T and Δ and is a quadratic function of $\nabla_j \Delta_i$ of the type first derived by De Gennes [3], and calculated microscopically by Cross [9]. For an explicit expression of ε_0 we take all 9 independent components of $\nabla_j \Delta_i''$, $\nabla_j \Delta_i''$ (taking into account the relations (6)) and keep the quadratic terms in the gradients $\nabla_j \Delta_i$ which are invariant with respect to an overall spatially uniform phase change of Δ . There are seven terms [4]

$$\begin{aligned} \varepsilon_0 = & a_1 |\Delta_j \nabla_j \Delta_i \Delta_i^*|^2 + |a_2 \Delta_j \nabla_j \Delta_i \Delta_i^* + \\ & + a_3 l_j \nabla_j \Delta_i l_i|^2 + a_4 |l_j \nabla_j \Delta_i \Delta_i^*|^2 \\ & + |a_5 l_j \nabla_j \Delta_i \Delta_i^* + a_6 \Delta_j^* \nabla_j \Delta_i l_i|^2 \\ & + a_7 |l_j \nabla_j \Delta_i l_i|^2 \quad (a_i \text{ reals}) . \end{aligned} \quad (11)$$

There are many different forms for ε_0 [3]. We remark that by partial integrations, i.e. on adopting a different choice of bulk and surface energy, we can change ε_0 in a non-trivial manner. Liu [10] has taken ε_0 as invariant with respect to partial integrations ; however, because he uses the ambiguous phase φ , we must take care. Partial integrations are not as simple as it appears with the notation $\nabla\varphi$ [4], and in calculating the local current density [9], the exact expression of ε_0 is very important.

The gauge invariance gives a very useful relation by writing that ε_0 does not change in a spatially uniform phase change of Δ (for example) i.e. a uniform rotation of the bivector $(\hat{\Delta}', \hat{\Delta}'')$ around $\hat{\mathbf{l}} = \frac{i}{2} \hat{\Delta} \wedge \hat{\Delta}^* :$

$$\Delta_2 = \Delta_1 e^{i\delta\varphi} \simeq \Delta_1 + i\Delta_1 \delta\varphi \quad \text{with} \quad \nabla\delta\varphi = 0 .$$

This condition $d\varepsilon_0 = 0$ gives :

$$i(h_{ij}^* \nabla_j \Delta_i - h_{ij} \nabla_j \Delta_i^*) = 0 . \quad (12)$$

The relation (12) is evident in an isotropic superfluid [6] where ψ replaces Δ : $h_j \nabla_j \psi - h_j \nabla_j \psi = 0$ because in this case $h_j = \frac{\rho S}{|\psi|^2} \nabla_j \psi$. The relation (12) can also be derived from the expression of

$$h_{ij} = \frac{\partial \varepsilon_0}{\partial (\nabla_j \Delta_i)}$$

together with relation (11).

If we write the general expression

$$\dot{\Delta} = A\Delta + B\Delta^* + C\mathbf{l}$$

the orthonormality conditions (6) lead to the form

$$\dot{\Delta} = ia\Delta + c\hat{\mathbf{l}} \quad (13)$$

where a is a real number and c is complex, which are functions of the thermodynamic variable ρ , $\mathbf{g}$, $\nabla_j \Delta_{ij}$, V^n , T . In contrast, to $\hat{\mathbf{l}}$, Δ transforms under time

reservoir like the order parameter ψ of an isotropic superfluid (or a wave function) i.e. :

$$\text{in } t \rightarrow -t : \Delta \rightarrow \Delta^* \quad \text{and} \quad \hat{\mathbf{I}} = \frac{i}{2} \Delta \wedge \Delta^* \rightarrow -\mathbf{I}. \quad (14)$$

We remark that $\hat{\Delta}'$ is even and $\hat{\Delta}''$ odd under time reversal.

In accordance with Graham, we insert equations (9) into equations (10), and we remark that unlike the derivative properties of the phase φ used by Graham, the time and space derivative of Δ commute so that :

$$T \frac{d(\rho S)}{dt} = -\nabla_i j_{\varepsilon i} + \mu \nabla_i g_i + V_i^n \nabla_j \sigma_{ij} - h_{ij}^* \nabla_j \dot{\Delta}_i - h_{ij}^* \nabla_j \dot{\Delta}_i^*. \quad (15)$$

First we calculate the reversible part of the equations. We insert (13) into (15) :

$$T \frac{d(\rho S)}{dt} = -\nabla_i j_{\varepsilon} + \mu \nabla_i g_i + V_i^n \nabla_j \sigma_{ij} - i h_{ij}^* \nabla_j (a \Delta_i) + i h_{ij} \nabla_j (a \Delta_i^*) - h_{ij}^* \nabla_j (c l_i) - h_{ij} \nabla_j (c^* l_i) \quad (16)$$

On the reversible level this entropy production after integration over space must reduce to zero apart from a surface contribution to the energy or entropy. Thus the relation (16) must reduce to a purely divergence term describing the flux of entropy, coming from the surface; this reduces severely the possible forms of the currents. First for $V^n = 0$, separating a in two arbitrary parts $a = a_1 + a_2$ we have

$$T \frac{d(\rho S)}{dt} = -\nabla_i j_{\varepsilon i} + \mu \nabla_j g_j - i a_1 (h_{ij_0}^* \nabla_j \Delta_i - h_{ij_0} \nabla_j \Delta_i^*) - i (\nabla_j a_1) (h_{ij_0}^* \Delta_i - h_{ij_0} \Delta_i^*) + (\nabla_j h_{ij_0}^*) c l_i + (\nabla_j h_{ij_0}) c^* l_i + i a_2 [(\nabla_j h_{ij_0}^*) \Delta_i - (\nabla_j h_{ij_0}) \Delta_i^*] - \nabla_j [h_{ij_0}^* (i a_2 \Delta_i + c l_i) + h_{ij_0} (-i a_2 \Delta_i + c^* l_i)]. \quad (17)$$

All these terms can be reduced to a divergence only if all the following relations are satisfied

$$\begin{aligned} \mu &= a_1 \theta^{-1} & g_{j_0} &= \theta i (h_{ij_0}^* \Delta_i - h_{ij_0} \Delta_i^*) \\ b &= i \beta (\nabla_j h_{ij}) l_i \end{aligned} \quad (18)$$

$a_2 = 0$ and $h_{ij_0}^*, \nabla \Delta_i - h_{ij_0} \nabla \Delta_i^* = 0$ which is the relation (12) already obtained. θ, β, a_1 are real numbers. The expression of superfluid current $\mathbf{g}_0$ is identical with the usual formula of superfluid current at $V^n = 0$

$$\mathbf{J}_s = \mathbf{g}_0 = \left(\frac{\partial \varepsilon_0}{\partial \mathbf{u}} \right) \frac{2m}{\hbar} = i \frac{2m}{\hbar} (h_{ij_0}^* \Delta_i - h_{ij_0} \Delta_i^*)$$

which gives

$$\theta = \frac{2m}{\hbar} \quad (19)$$

$\mathbf{u}$ is the fictitious potential [3] vector which appears in relation (8) and in the energy ε_0 when we replace ∇

by $(\nabla - i\mathbf{u})$. The relation $a = \frac{2m}{\hbar} \mu$ is equivalent

to remark that the phase φ of superfluid (ambiguous local phase) is conjugate to the particle number.

For fixed orientation of $\mathbf{I}$ the equation $\dot{\Delta} = i \frac{2m}{\hbar} \mu \Delta$

(when $\nabla \Delta = 0$) is the generalization of the isotropic superfluid Josephson equation $\dot{\varphi} = \frac{2m}{\hbar} \mu$; this is

more evident if we write this last equation with the superfluid wave function $\psi, \dot{\psi} = i \frac{2m}{\hbar} \mu \psi$. This shows

that the generalization of ψ to Δ is very simple. The divergence gives $\mathbf{j}_{\varepsilon_0} = \mu \mathbf{g}_0$ if we neglect the second-order term in h_{ij} . We note that in the resulting equations the parity conditions are satisfied.

When $V^n \neq 0$ a Galilean transformation gives for linear term in V^n and $\nabla_j \Delta$ first : (we note with the subscript 0 the frame where $V^n = 0$)

$$\mathbf{g} = \mathbf{g}_0 + \rho V^n \quad \text{and} \quad \mu = \mu_0 \quad (20)$$

and

$$j_{\varepsilon} = \mu \mathbf{g} + T \rho S V^n \quad \text{and} \quad \varepsilon = \varepsilon_0 + V^n \mathbf{g}_0 + \rho \frac{V_n^2}{2}. \quad (21)$$

In a Galilean transformation Δ is transformed like an isotropic superfluid wavefunction ψ , i.e. it changes according to the relation (7) :

$$\Delta(\mathbf{r}) = \Delta_0(\mathbf{r}) \exp \left[i \frac{2m}{\hbar} \varphi(\mathbf{r}) \right]$$

with

$$\varphi(\mathbf{r}) = V^n \cdot \mathbf{r} + \frac{1}{2} V_n^2 t \sim V^n \cdot \mathbf{r}$$

(we keep only linear terms). If we call the elastic energy in the frame at velocity V^n

$$\varepsilon^{\text{Elast.}} = \varepsilon_0(\nabla_j \Delta) = \varepsilon_0 \left[\nabla_j \left(\Delta_0 \exp \left[i \frac{2m}{\hbar} \varphi \right] \right) \right]$$

we have

$$\varepsilon = \varepsilon^{\text{Elast.}} + V_i^n \bar{\rho}_{ij}^n V_j^n = \varepsilon_0(\nabla_j \Delta_0) + V^n g_0 + \rho \frac{V_n^2}{2} \quad (22)$$

which is equivalent to the definition of the normal fluid density tensor

$$\bar{\rho}_{ij}^n = \rho_{\parallel}^n l_i l_j + \rho_{\perp}^n (\delta_{ij} - l_i l_j).$$

By definition of the Gibbs relation (10) we have :

$$h_{ij} = \frac{\partial \varepsilon_0}{\partial (\nabla_j \Delta_i^*)} = \left(h_{ij_0} + V_0^n \frac{\partial \mathbf{g}_0}{\partial (\nabla_j \Delta_{i_0}^*)} \right) \times \exp \left[i \frac{2m}{\hbar} \varphi(\mathbf{r}) \right]. \quad (22')$$

When $\nabla_j V_i^n$ is also different from zero we have new terms in Δ (i.e. in a and c of the relation (13)) linear in $\Delta_j V_i^n$. If we write as in formula (13)

$$\dot{\Delta}_i = (\alpha_{jk} \Delta_j + \gamma_{jk} l_j) \nabla_j V_k^n \quad (23)$$

the condition (6) gives

$$\alpha_{jk} = -\alpha_{kj}^* \quad (24)$$

and the parity with respect to time gives

$$\alpha_{jk}(t) = \alpha_{jk}^*(-t) \quad \text{and} \quad \gamma_{jk}(t) = -\gamma_{jk}^*(-t). \quad (25)$$

Eq. (23) shows that in a gauge transformation α_{jk} is invariant and γ_{jk} varies like Δ . The most general form compatible with all these conditions is :

$$\begin{aligned} \alpha_{jk} &= ib_1 l_r \varepsilon_{rjk} + b_2 (\Delta_j \Delta_k^* - \Delta_j^* \Delta_k) \\ \gamma_{jk} &= ib_3 \Delta_r \varepsilon_{rjk} + b_4 l_j \Delta_k + b_5 l_k \Delta_j \end{aligned} \quad (26)$$

where b_1 to b_4 are real numbers.

Physically, as in nematics, and as remarked by Liu [10] we add the hypothesis that in a rotating vessel, where $\mathbf{V}^n = \boldsymbol{\omega} \wedge \mathbf{r}$ and $\boldsymbol{\omega} = \frac{1}{2} \text{rot } \mathbf{V}^n$, the superfluid is in equilibrium with the rotating normal fluid (except for surface effects). As in liquid crystal it can be closely related [13] to the conservation of total angular momentum and to the conservation of the elastic energy ε_0 with respect to rotation of both $\mathbf{r}$ and $\Delta(\mathbf{r})$ [5, 11]. In this case, collecting the preceding results we obtain for the reversible evolution :

$$\dot{\Delta}^R = \frac{1}{2} \text{rot } \mathbf{V}^n \wedge \Delta + i \frac{2m}{\hbar} \mu \Delta + i\beta (\nabla_j h_{ij}) l_i \hat{\mathbf{I}} + \alpha (l_j \Delta_k + l_k \Delta_j) (\nabla_j V_k^n + \nabla_k V_j^n) \hat{\mathbf{I}}. \quad (27)$$

This equation, locally identical to the relation (7) of Graham [1], describes both the movement of $\hat{\mathbf{I}}$ and of $\hat{\Delta}'$, $\hat{\Delta}''$ without the ambiguity of the phase φ . Following Graham, the term having coefficient β describes a precession of $\hat{\mathbf{I}}$ around the effective field $H_i = \nabla_j h_{ij}$, but we also give the movement of the vectors $\hat{\Delta}'$ and $\hat{\Delta}''$ perpendicular to $\hat{\mathbf{I}}$ during this precession, which, when using the phase φ , is quite ambiguous for large angles as explained. We remark that an electric field or the spin-orbit coupling can also create a uniform field $\mathbf{H}_i = \delta\varepsilon_0/\delta\Delta_i$ when there are no gradients of the order parameter Δ . The stress tensor α_{ij}^R is easily fixed by the condition of zero entropy production : when we put relation (27) into relation (15) :

$$\begin{aligned} \sigma_{kj}^R &= -\alpha \{ H_i l_i (l_j \Delta_k + l_k \Delta_j) + H_i^* l_i (l_j \Delta_k^* + l_k \Delta_j^*) \} + \\ &+ \frac{1}{2} (H_i \varepsilon_{irs} \varepsilon_{rjk} \Delta_s + H_i^* \varepsilon_{irs} \varepsilon_{rjk} \Delta_s^*) + \rho \delta_{ik} + \varphi_{kjml} \nabla V^n \end{aligned} \quad (28)$$

φ_{ijk} has been introduced by Liu [10] as a solution of $\nabla_j V_i^n \varphi_{ijkl} \nabla_k V_l^n = 0$ i.e.

$$\varphi_{ijkl} = \left\{ (\varphi_1 (\delta_{ik} - l_i l_k) + \varphi_2 l_i l_k) l_p \varepsilon_{pjl} + \begin{pmatrix} i \rightarrow j \\ k \rightarrow k \\ j \rightarrow i \\ l \rightarrow l \end{pmatrix} + \begin{pmatrix} i \rightarrow i \\ k \rightarrow l \\ j \rightarrow j \\ l \rightarrow k \end{pmatrix} + \begin{pmatrix} i \rightarrow j \\ k \rightarrow l \\ j \rightarrow i \\ l \rightarrow k \end{pmatrix} \right\}.$$

The dissipative parts of the equations are obtained under the constraints of positive entropy production, correct space and time symmetry and the Onsager relations. For $\dot{\Delta}^D$ the method is similar to the derivation of equations (23), (24), (25), (26) except for the inverse parity in equations (25). $\dot{\Delta}^D$ is linear in H_i and $\nabla_j V_k^n$. At first the reversible coefficients become simply imaginary and secondly it appears new terms : (with $H_i = \nabla_j h_{ij}$)

$$\begin{aligned} \dot{\Delta}^D &= -\zeta_3 (H_j \Delta_j^* - H_j^* \Delta_j) \Delta + \beta^D (l_j H_j) \hat{\mathbf{I}} + \\ &+ \frac{i}{2} \{ (\zeta_1 \delta_{jk} + \gamma^D l_j l_k) \Delta + \alpha^D (l_j \Delta_k + l_k \Delta_j) \hat{\mathbf{I}} \} (\nabla_j V_k^n + \nabla_k V_j^n) \quad (\zeta_3 > 0 \quad \beta^D > 0). \end{aligned} \quad (29)$$

The symmetry of the Onsager coefficients and positive entropy production fixes the stress tensor and the energy flux :

$$\begin{aligned} T \frac{d}{dt} \int_v S dv &= \int dV \left\{ -\frac{\nabla_i T}{T} j_{ei}^D - \nabla_i V^n \sigma_{ji}^D + H_i^* \dot{\Delta}_i^D + H_i \dot{\Delta}_i^D \right\} \\ j_r^D \varepsilon &= -\mathbf{K}_{rj} \nabla_j T \quad (\mathbf{K}_{rj} = K_{\parallel} l_r l_j + K_{\perp} (\delta_{rj} - l_r l_j); K_{\parallel}, K_{\perp} \geq 0) \end{aligned}$$

$$\begin{aligned} \sigma_{kj}^D &= -v_{kjl} \frac{1}{2} (\nabla_i V_l^n + \nabla_l V_i^n) + i(\gamma l_j l_k + \zeta_1 \delta_{jk}) (H_i^* \Delta_i - H_i \Delta_i^*) + \\ &+ i\alpha^D \{ H_i l_i (l_j \Delta_k + l_k \Delta_j) - H_i^* l_i (l_j \Delta_k^* + l_k \Delta_j^*) \}. \end{aligned} \quad (30)$$

These equations are physically identical to the equations of Graham except for the ambiguity of phase φ which is completely removed. When $\hat{\mathbf{l}}$ is uniform in space, $\Delta = \Delta_0 e^{i\varphi}$ has a well-defined phase and we get the hydrodynamic equations similar to the Khalatnikov's [6] superfluid equations, if we take into account the uniform anisotropy; ζ_1, ζ_3 are identical to the second viscosity, γ is an additional coefficient to ζ_1 introduced by the anisotropy; this identification comes from the remark : $\text{div } \mathbf{g}_0 = i(H_i^* \Delta_i - H_i \Delta_i^*)$ taking into account the relations (12) and (19).

The five normal viscosity terms v_{ijk} are given by the same formulae as in nematic liquid crystals if we take as director $\hat{\mathbf{l}}$; this includes as a special case the first η and second ζ_2 viscosity coefficient of isotropic superfluid [6].

$$\sigma_{ij}^v = v_{ijkl} A_{kl} = \eta A_{ij} + v_2 (l_r l_j A_{ri} + l_r l_i A_{jr}) + v_3 l_i l_j l_k l_l A_{kl} + \zeta_2 \delta_{ij} A_{ll} + v_5 l_i l_j A_{ll} + v_6 \delta_{ij} l_k l_l A_{kl}, \quad (31)$$

$$r = \eta + 2v_2 + v_3 + \zeta_2 + v_5 + v_6 \geq 0; \quad \eta + \zeta_2 \geq 0; \quad \eta + v_2 \geq 0; \quad \eta \geq 0; \quad \zeta_1^2 \leq 4r\zeta_3; \\ \gamma^2 \leq 4r\zeta_3; \quad (v_5 + v_6)^2 \leq 4r(\eta + \zeta_2) \quad \text{if } \zeta_2 < 0, \quad \eta \geq -2\zeta_2, \quad \zeta_1^2 \leq 4(\eta + \zeta_2)\zeta_3.$$

β^D , similar to the coefficient η of Graham, gives the dissipation associated with the movement of $\hat{\mathbf{l}}$ under the field $\mathbf{H}$. α^D similar to ζ of Graham gives the movement of Δ under the influence of a gradient of V^n . We remark that contrary to liquid crystals, it is possible to calculate from the microscopic theory the preceding hydrodynamic coefficients [9]. For uniform $\hat{\mathbf{l}}$, these equations give the same physical results as Graham's equations for first, second, fourth round and orbital waves.

These equations are identical at each point to that of reference [1] to first order. But for practical applications with small velocities we need to integrate only these equations without compatibility equations. The very complex non linear compatibility equations [4, 8] which must be added to the linear equations of reference [1] when $\hat{\mathbf{l}}$ wanders slowly through large angles, dissuades to undertake practical calculations similar to those which have been made in liquid crystals.

One simple application is the equilibrium orientation of Δ in a shear flow of normal velocity. The problem is more complicated than in a nematic liquid crystal, because we have a bivector. For this reason we do not find, after calculations, any equilibrium position of the bivector in a shear flow for all values of the hydrodynamic coefficients, except for singular ones. This shows that any small normal velocity gradient always induces a distortion of Δ

and $\mathbf{l}$ in the A phase. This remark can help to explain the low critical currents for heat conduction [12] observed in the A phase.

Finally we have proposed hydrodynamic equations for the orbital part of the A phase without using the phase φ , which is not defined. We describe the A phase by the true complex order parameter Δ which is similar both to the director $\mathbf{n}$ of a nematic liquid crystal and to the order parameter ψ of a superfluid. These equations can be the starting point for phenomenological equations of textures in the A phase, (except core structures) taking into account the irreversible effects such as normal flow or temperature gradients. The equations being more complex than in nematics, we can expect that the instabilities in a normal fluid shear flow are different. As found first by Toulouse and Kléman [14] the topological structure of defects of the true A phase is completely different from a hypothetical anisotropic phase where φ would be well defined. In order to simplify as much as possible the calculations for the hydrodynamics and free energy minimization, it seems necessary to avoid any reference to a phase φ and thus it is better to use the true order parameter as done in this paper.

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