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ON THE SUPERFLUID PHASES OF $^3$He IN A MAGNETIC FIELD

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Abstract. — The behaviour of superfluid $^3$He in the presence of a magnetic field is discussed assuming an $L = 1$ pairing. In particular we predict a new phase with a spatially non uniform (sinusoidal) order parameter. These results are extended in order to take into account strong-coupling effects and we show how measurements under a magnetic field may provide information on the phenomenological Landau coefficients.

A complete experimental study of the effect of a magnetic field on the superfluid phases of $^3$He would be a good test of the p-wave assumption which is commonly accepted. This effect has been theoretically investigated by Ambegaokar and Mermin [1], within the framework of the weak-coupling model, and for small values of the field. Takagi [2] has studied the general case for the $A_1$ and $A_2$ phases. We shall examine the problem with less restrictive assumptions: we first consider the weak-coupling model for arbitrary fields; in the second part we show how the weak-coupling results have to be modified to take into account strong-coupling effects in all the phases, in the limit of small fields.

1. The weak-coupling model. — The general form of the condensation amplitudes in the different phases may be seen from symmetry considerations alone. The invariance group of the normal phase is generated by the operators $N_1$, $N_{-1}$, $L$ [11]. If we assume, following Ambegaokar and Mermin, that a transition to a state with up spin pairs will take place first, the order parameter in the $A_1$ phase is of the form

$$< a_{-k_1} a_{k_1} > \propto Y^m_l(\hat{k}).$$

In the weak-coupling theory $m = \pm 1$ are the stable solutions. If, for instance, $m = + 1$, one easily sees that the invariance group of the $A_1$ phase is generated by $N_1$ and $-\frac{1}{2} N_1 + h^{-1} L_z$. Then the order parameter of the second order transition to the $A_2$ phase which breaks this invariance is

$$< a_{-k_1} a_{k_1} > \propto Y^\pm_1(\hat{k}).$$

In the strictly weak-coupling model the solutions $Y^\pm_1$ are degenerate; a small deviation from this model stabilizes either $Y^+_1$ (ABM like state) or $Y^-_1$ (planar like state). These phases are still invariant under the change $k_z \to -k_z$. If a second order transition may occur to a phase with $S_z = 0$ pairs, then the corresponding order parameter is

$$< a_{-k_1} a_{k_1} > \propto \hat{k}_z.$$

Ambegaokar and Mermin have shown that such a transition is expected for fields $H > H_1 (\approx 1$ kG). For $H < H_1$ the transition becomes first order. However, several years ago Fulde and Ferrell [3] suggested that a conductor might go into a superconducting state in which the Cooper pairs have a non zero total momentum, when an exchange field acts on the spins of the conduction electrons; in such a state the translation invariance is also broken. Sarma and Saint-James [4] have shown that the Fulde-Ferrell state could only exist in the case of clean superconductors, i.e. when the motion of the electrons is not affected by impurity scattering. The transition to the normal state is second order. Such a situation...
is expected in $^3$He, which can be viewed as a clean superfluid [9], with an order parameter:

$$\langle a_{-k+q/21} a_{k+q/21} \rangle \propto \hat{k}_z \delta(q \pm Q).$$

Therefore we shall examine the antiparallel pairing for high values of the field. Allowing for the degeneracy of the axial and planar solutions, we take the gap parameters in the A$_2$ phase as

$$A_0(k, q) = T \sum_{\pi} V_{\pi k} \left[ G_1 \left( k' + \frac{q}{2}, - \omega_\pi \right) G_1 \left( - k' + \frac{q}{2}, - \omega_\pi \right) A_0(k', q) + F_1 \left( k' + \frac{q}{2}, \omega_\pi \right) F_1 \left( - k' + \frac{q}{2}, - \omega_\pi \right) A_0^*(k', - q) \right]$$

where $G_\pi$ and $F_\pi$ are the single particle propagators in the A$_2$ phase [8].

Performing the frequency sum, one gets:

The solution $H(T)$ of this implicit equation has been numerically computed after solving the gap equation for $A$ and is plotted in figure 1. It exhibits a smooth bump as in the case of superconductors [10]; this suggests an instability of the second order transition, as will be shown in what follows.

1.1 THE A$_2$-B TRANSITION. — The solution of (1) with $q = 0$ yields the A$_2$-B line, given in reference [1] near $T_c$ within the Landau theory:

$$1 = v T \sum_{\pi} \sum_k \hat{k}_z^2 \frac{\epsilon^2}{(\omega_\pi^2 + \epsilon^2 + |A(k)|^2)}.$$  

Performing the frequency sum, one gets:

$$\frac{1}{vN(0)} = \int_{-\omega_c}^{+\omega_c} \frac{d\epsilon}{4 \epsilon} \int_0^{\pi} \sin \theta \cos^2 \theta \times$$

$$\times \left[ \frac{\epsilon + \gamma H}{(\epsilon + \gamma H)^2 + A^2 \sin^2 \theta} \right]^{1/2} \sin^2 \theta \left[ \frac{\epsilon + \gamma H}{(\epsilon + \gamma H)^2 + A^2 \sin^2 \theta} \right]^{1/2}.$$  

The solution $H(T)$ of this implicit equation has been numerically computed after solving the gap equation for $A$ and is plotted in figure 1. It exhibits a smooth bump as in the case of superconductors [10]; this suggests an instability of the second order transition, as will be shown in what follows.

1.2 THE SINUSOIDAL STATE. — The physical explanation for a Fulde-Ferrell state in $^3$He is the following (at $T = 0$): in the presence of an external magnetic field the Fermi surface splits into 2 spheres $S_1$ and $S_2$; in the $A_2$ phase the pairing takes place independently on each of these surfaces and mostly involves the particles lying in the $xy$ plane since

$$|A_\pm|^2 \propto \hat{k}_x^2 + \hat{k}_y^2.$$  

Therefore the region along the $z$ axis is available for antiparallel pairing. Figure 2 shows the usual BCS pairing; but in order to associate a particle on $S_1$ with the gives the highest critical temperature for the antiparallel pairing. The linearized equation for the gap $A_0$ is written as
with one on $S_1$ one may form a pair with a non-vanishing center of mass momentum along the $z$ direction, i. e. parallel to the angular momentum (Fig. 3). For sufficiently high fields such a phase is expected to have a lower energy. Moreover in the second case the solid angle containing the particles to be paired is smaller than in the BCS case and the corresponding transition would not be so strongly affected by the $A_z$ components.

FIG. 3. — Antiparallel pairing with non zero total momentum.

Therefore we look for solutions of (1) of the form

$$A_0(k, Q) = A_0^*(k, -Q) \propto \hat{k}_z.$$ 

The Fulde-Ferrell state is favorable when

$$K(q, \Delta) > K(0, \Delta)$$

where

$$K(q, \Delta) = \frac{vT}{\pi} \sum_k \hat{k}_z^2 \left[ G_1\left(k + \frac{q}{2}, \omega_n\right) \times \right.$$

$$\times G_1\left(-k + \frac{q}{2}, -\omega_n\right) + F_1\left(k + \frac{q}{2}, \omega_n\right) \times$$

$$\left. + F_1\left(-k + \frac{q}{2}, -\omega_n\right)\right].$$

We may write

$$K(q, \Delta) = K(0, \Delta) + [K(q, \Delta) - K(0, \Delta)].$$

The term between brackets does not depend strongly on $\Delta$ as previously seen [5] so that we expand it up to second order in $\Delta$

$$K(q, \Delta) - K(0, \Delta) \approx K(q, 0) - K(0, 0)$$

terms proportional to $|\Delta(q)|^2$.

The transition point is first reached for $|\Delta(q)|^2 = 0$, i. e. $q \parallel O_z$.

The kernel calculated in the normal phase is

$$K(q, 0) = \frac{vT}{\pi} \sum_k \hat{k}_z^2 \frac{1}{\varepsilon + \frac{q^2}{2}} - i\omega_n \frac{1}{\varepsilon - \frac{q^2}{2}} + i\omega_n.$$ 

Taking $\varepsilon_{k+q/2} \approx \varepsilon_k + q \cdot v$ as is usually done [8]

$$K(q, 0) = 2\pi vTN(0) \times$$

$$\times \text{Re} \sum_{\omega > 0} \int_0^{\pi} \frac{\cos^2 \theta \sin \theta d\theta}{\omega + i(qv \cos \theta + \gamma H)}.$$ 

The curvature $\frac{\partial^2 K}{\partial q^2}(q = 0)$ becomes positive for $\gamma H/kT > 1.911$ as shown in reference [4]. The Fulde-Ferrell state is stable if $T < T^* \approx 0.45 T_c$ and if $\gamma H > \gamma H^* \approx 0.85 T_c$, $T^*$ and $H^*$ being the coordinates of the intersection point of the straight line $\gamma H/kT = 1.911$ and of the curve $H(T)$ computed in 1.1. The transition from the F-F state to the uniform (BW) state is first order. The sinusoidal phase has been represented in figure 1 (however the $A_2$-$A_1$ and $A_1$-FF lines have not been calculated exactly. In summary this state is characterized by a longitudinal wave

$$A_0(k, x) = A_0 \cos(Ql \cdot x) \hat{k}_x.$$ 

where $l$ is a unit vector along the direction of the angular momentum, thus perpendicular to the field $H$, and $Q$ is a function of $H$ and corresponds to the maximum of $K(q, 0) : \delta K/\delta q(Q, 0) = 0$. From an experimental point of view, the two lines $A_2$-$FF$ and FF-$B$ could be deduced from specific heat measurements. Evidences for the sinusoidal state would be given by neutron diffraction or NMR since a spin density wave is associated with the gap (2), or by acoustic attenuation measurements. In a thin film experiment, with a field parallel to the film, one may observe a resonance...
for the values of the field yielding a wavelength commensurate with the thickness of the film \(^{(1)}\).

2. Extension of the weak-coupling model. — We restrict ourselves to the case of weak fields (i.e. close to \(T_c\)). The free energy reads:

\[
\mathcal{F} = \frac{3}{2} \int d\Omega(k) \times \\
\left[ x_+ |A_+(k)|^2 + x_- |A_-(k)|^2 + \alpha_0 |A_0(k)|^2 \right] + \mathcal{F}^{(4)}.
\]

We take for the fourth-order contribution \(\mathcal{F}^{(4)}\) the zero field terms of reference [6]:

\[
\mathcal{F}^{(4)} = \beta_1 \left| \text{tr} \, AA^+ \right|^2 + \beta_2 \left( \text{tr} \, AA^+ \right)^2 + \\
+ \beta_3 \text{tr} \left[ A \tilde{A} (A \tilde{A})^* \right] + \beta_4 \left( \text{tr} \, AA^+ \right)^2 + \\
+ \beta_5 \text{tr} \left[ A \tilde{A} (A \tilde{A})^* \right].
\]

The matrix \(A\) is defined by

\[
\begin{pmatrix}
A_+ & A_0 \\
A_0 & A_-
\end{pmatrix} = \sum_{\nu} (\sigma_\nu, A_\nu, \tilde{k}_\nu) i\sigma_y.
\]

We have to consider the 3 solutions (using continuity arguments with the weak-coupling case):

axial (ABM):

\[
A_+(k) = A_+ (-\tilde{k}_x + i\tilde{k}_y), \\
A_-(k) = A_- (-\tilde{k}_x + i\tilde{k}_y) \quad A_0(k) = 0
\]

planar (P):

\[
A_+(k) = A_+ (-\tilde{k}_x + i\tilde{k}_y), \\
A_-(k) = A_- (\tilde{k}_x + i\tilde{k}_y) \quad A_0(k) = 0
\]

BW:

\[
A_+(k) = A_+ (-\tilde{k}_x + i\tilde{k}_y), \\
A_-(k) = A_- (\tilde{k}_x + i\tilde{k}_y) \quad A_0(k) = A_0 \tilde{k}_z.
\]

The corresponding \(\mathcal{F}^{(4)}\) energies are

\[
\mathcal{F}^{(4)}_{\text{ABM}} = (\beta_2 + \beta_3) (A_+^2 + A_-^2) + \\
+ 2(\beta_2 + \beta_3 + 2\beta_4) A_+^2 A_-^2 + \\
\mathcal{F}^{(4)}_{\text{P}} = (\beta_2 + \beta_4) (A_+^2 + A_-^2) + \\
+ 2(2\beta_1 + \beta_2 + \beta_3 + \beta_5) A_+^2 A_-^2 + \\
\mathcal{F}^{(4)}_{\text{BW}} = (\beta_2 + \beta_4) (A_+^2 + A_-^2) + \\
+ 2(2\beta_1 + \beta_2 + \beta_3 + \beta_5) A_+^2 A_-^2 + \beta_6 A_0^4 + \\
+ 2\beta_2 (A_0^2 A_+^2 + A_0^2 A_-^2) + 4\beta_1 A_0^2 A_+ A_-.
\]

(we set \(\beta_0 = \sum_{i=1}^{5} \beta_i\)).

We have seen that the axial and planar solutions are degenerated in the weak-coupling limit. More generally this degeneracy occurs when

\[
2\beta_1 + \beta_3 - \beta_4 - \beta_5 = 0.
\]

The hyperplane \((\Pi)\) defined by (4) divides the space of the parameters \(\beta_i\) into 2 regions. If

\[
2\beta_1 + \beta_3 - \beta_4 - \beta_5 > 0
\]

(region I) the axial state has the lower energy ; in this case a continuous change to the B type phase is impossible and the transition becomes first order (Fig. 4).

If \(2\beta_1 + \beta_3 - \beta_4 - \beta_5 < 0\) (region II) the planar state is stable and the transition is still second order. The behaviour of the phase diagram is thus different according to whether the physical manifold

\[
\beta_i = \beta(T,P)
\]
cuts the hyperplane \((\Pi)\) or not. This is shown in figures 5 and 6.

The same arguments hold for the transition to the sinusoidal state : it is first order in region I, second order in region II. However the Fulde-Ferrell state is expected to vanish at high pressure, when spin fluctua-
Fig. 6. — a), b), c) Evolution of the phase diagram $P, T$ at increasing applied field if the manifold $f(T, P)$ crosses (II).

The specific heat jump given in reference [1] is not valid and actually diverges for all values of the field because of the fluctuations of the order parameter in the set of the degenerated solutions (the degeneracy was arbitrarily removed in reference [1]). From the previous considerations, one may derive some quantitative results for the first order transition (for weak fields).

From (3) one gets the energy of the ABM state

$$\mathcal{F}_{ABM} = \frac{\alpha^2}{2} \left( -\frac{t^2}{B_1} + \frac{\eta^2 h^2}{2 \beta_s} \right)$$

and the energy of the BW state assuming a small $\Delta_0$

$$\mathcal{F}_{BW} = \frac{\alpha^2}{2} \left( -\frac{t^2}{B_2} + \frac{\eta^2 h^2}{B_3} \right) + \frac{\beta_0^2}{\beta_s},$$

where we have set $\alpha = \alpha(t \pm \eta h)$ following reference [1],

$$B_1 = 2(\beta_2 + \beta_4 + \beta_5),$$

$$B_2 = 2 \beta_1 + 2 \beta_2 + \beta_3 + \beta_4 + \beta_5,$$

$$B_3 = 2 \beta_1 + \beta_3 - \beta_4 + \beta_5,$$

$$\alpha_0(t, h) = \alpha_0(t, h) - \frac{2 \alpha \beta_2 t}{B_2} - 2 \alpha \beta_4 \left( \frac{t^2}{B_2} - \frac{\eta^2 h^2}{B_3} \right)^{1/2}$$

$\alpha_0(t, h)$ is of the form $\alpha t - \beta h^2$.

The parameters $\alpha$, $\eta$, $B_1$ and $\beta_5$ can be fitted with measurements on the $A_1$ and $A_2$ transitions [2], [7]. The other parameters $b$, $\beta_1$, $\beta_2$, $\beta_3$, $\beta_4$ may be obtained from the shape of the $A_2$-B line whose equation is given by

$$\mathcal{F}_{ABM} = \mathcal{F}_{BW}$$

and from the entropy jump

$$\left( -\frac{\partial F_{BW}}{\partial T} + \frac{\partial F_{ABM}}{\partial T} \right),$$

magnetic susceptibility, etc... The validity of these formulas is restricted to low pressure, i.e. when the deviation from the weak-coupling model is small (one may guess $P < P_{pcp} \approx 20$ bar).

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References

[5] This can be seen on the exact expression of $K(q, d)$ after performing the energy integration.
[9] The collisions between quasi-particles are negligible since the mean free path is much greater than the coherence length.