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SOME RESULTS OF THE BONNET TRANSFORMATION

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Introduction

No in-depth knowledge of differential geometry is needed to appreciate the special properties of the Bonnet transformation, but an understanding of the concept of curvature is necessary. With the aid of that tool, it is possible to appreciate the very special role of this very restrictive surface transformation.

Curvature

The curvature of a circle is simply the inverse of the radius of the circle. This intuitively makes sense; a straight line is a circle with infinite radius and hence zero curvature while a small circle has a small radius and a large curvature. At any (regular) point a space-curve can be approximated by a circle and the curvature at that point is simply the curvature of the circle that best approximates the curve at that point. The circle of curvature defines a plane and as we move along that curve, subsequent circles of curvature will recede from the original plane. The rate of recession is the torsion of the curve. If the torsion is zero, the rate of recession from the plane is zero and the curve is planar. The curvature of a space-curve is denoted by $K$ and the torsion by $\tau$.

On a surface the concept of curvature is a little more involved, but it can be studied by means of curves on the surface.

If we consider a point on a surface and the surface normal through this point, it is clear that we may draw a plane, containing this normal and intersecting the surface. The line of intersection between the plane and the surface. If the plane is rotated around the normal, the value of the normal curvature will change in a cyclic fashion. The extremal values of the normal curvature are known as the principal curvatures of the surface at the point and they occur along the principal directions of the surface. The normal curvature is denoted by $k_n$ and the principal curvatures by $k_1$ and $k_2$ (conf fig 1).

Fig 1. Normal curvature on a surface
We are now ready to define the composite entities known as the Gaussian and the mean curvatures of a surface. The Gaussian curvature, $K$, is defined as the product of $k_1$ and $k_2$ and the mean curvature, $H$, as their arithmetic mean:

$$K = k_1 \ast k_2$$

$$H = (k_1 + k_2)/2$$

$K$ is a measure of the metric of the surface. When a surface is subjected to a transformation that leaves the metric intact (bending without stretching), $K$ will not change. An important characteristic of $H$ is that it changes sign when the surface changes orientation. When this is forbidden from symmetry considerations, $H$ must be identically equal to zero, and the surface is a minimal surface.

It is important to note that while knowledge of $K$ and $\tau$ is sufficient to completely determine a space-curve, $H$ and $K$ do not generally fix a surface. In the special case of minimal surfaces, fixing $H$ and $K$ restricts us not to a single surface, but to a family of surfaces. The transformation from one member of the family to another is known as the Bonnet transformation.

The Bonnet transformation

This transformation discovered by Ossian Bonnet (1) is highly restrictive. It is continuous and it preserves both the mean and the Gaussian curvature of every point on the surface. Such restrictive transformations are very unusual, but not unique to minimal surfaces. As we shall see, parallel surfaces to minimal surfaces also undergo Bonnet-like transformations. The perhaps most well known example of a Bonnet transformation is that between the helicoid and the catenoid (conf fig 2).

Fig 2. The Bonnet transformation from the catenoid to the helicoid
The Bonnet transformation is mathematically described as the weighted sum of two minimal surfaces:

\[ S' = \cos \Theta S^0 + \sin \Theta S^0 \]

where the points \( S^0 \) and \( S^0 \) vary on the given minimal surfaces and have equal normals. For each surface \( S' \), there is a unique \( S^0 \) such that \( S' \) is isometric to \( S^0 \). The surfaces \( S^0 \) and \( S^0 \) are then said to be adjoint and the family \( S' \) is called the Bonnet transformation of \( S^0 \). The surfaces \( S' \) are said to be associate to \( S^0 \). The angle \( \Theta \) is the association parameter, or simply the Bonnet angle.

Let us examine what makes the Bonnet transformation so special. First of all the Bonnet transformation is well defined. It can follow one path only. As a minimal surface undergoes the Bonnet transformation the individual points move along ellipses. Each point on the ellipse corresponds to a unique surface in the family. This puts the Bonnet transformation (and therefore the minimal surfaces) in a very special position. The sphere cannot be bent at all without changing \( K \). This is known as the rigidity of the sphere. The plane on the other hand has an infinite number of possible modes of isometric (\( K \)-preserving) deformations, something that might be called the floppyness of the plane. For a minimal surface there exists one such mode only if the minimal surface properties are to be preserved.

A further very special property can be read directly from the formulae determining \( K \) and \( H \) for parallel surfaces. A parallel surface is defined in terms of an original surface as

\[ y = x + Na \]

where \( x \) and \( y \) are the coordinates of the original and parallel surfaces, \( N \) is the unit surface normal and \( a \) is the distance between the two surfaces. It is easy to show that \( H \) and \( K \) for the parallel surface behave as

\[ K_s = K_s/(1-2H_s+K_sa^2) \]
\[ H_s = (H_s-aK_s)/(1-2H_s+K_sa^2) \]

For a minimal surface undergoing the Bonnet transformation, \( H \) and \( K \) are invariant, and therefore it is clear that this is true for parallel surfaces as well.

The Bonnet transformation preserves all local axes of rotation and rotoinversion normal to the surface (2). This determines what symmetries are possible for periodic minimal surfaces generated by the Bonnet transformation. Periodic minimal surfaces have a specific obstacle to overcome when undergoing the Bonnet transformation. The transformation is continuous only for surfaces of genus zero, that is for disc-like surfaces. When a surface of higher genus is transformed, it must be cut open. In the case of periodic minimal surfaces, these cuts close up at the Bonnet angles where the surface is once more periodic. Some important characteristics of the Bonnet transformation are thus:

* Is is isometric, i.e. it preserves distances along the surface.
* It is the only isometric transformation of a minimal surface through minimal surfaces
* The parallel surfaces undergo Bonnet-like transformations as well
* The Bonnet transformation preserves all axes of rotation and rotoinversion normal to the surface
* Surfaces that are not disc-like must be cut open to accommodate the Bonnet transformation
Biochemical applications

To the chemist, the Bonnet transformation is an attractive way to describe complex reorganisations. In biochemistry it is very common that macromolecules attain minimal surface shape, and for the kind of molecules that undergo drastic dynamic structure changes, it is easy to see that the Bonnet transformation must be very advantageous indeed. There will be a well defined, low energy path from one state to another. The transformation is isometric, and therefore no bonds are stretched, it will proceed along a well defined path since the isometry is unique and, perhaps most important in the case of biomolecules, the parallel surfaces undergo Bonnet-like transformations as well, leaving any hydration shell virtually unperturbed.

Inorganic applications.

The Bonnet transformation is equally powerful in describing some transitions in inorganic chemistry. One example of how the Bonnet transformation might be used to suggest a possible pathway for an inorganic structural transition is the description of the martensitic transition given by Hyde and Andersson (3). The martensitic transition takes place when austenite is quenched, forming martensite. The fcc lattice of the austenite is transformed into the bcc lattice of martensite. The characteristics of the transformation include structural change involving drastic rearrangements of atoms, but no diffusion, very high speed, tweeding and a precise orientational relationship. By putting the iron atoms of the austenite structure on the flat points of the D-surface and bending via the Bonnet transformation, many of these phenomena are explained. The lattice parameter changes by a factor 1.255 in the martensitic transition, and the corresponding change on bending the D-surface into the gyroid is 1.269. The Bonnet transformation preserves normal directions, and since the three atomic sites that define the (111) plane in austenite are transformed into sites that define the (011) plane in martensite, the orientational relationship is explained. Tweed formation is a necessary effect of a Bonnet transformation acting on a surface with a genus higher than zero, and the bending is simultaneous over the entire surface, something that will provide high speed.

Mathematical applications

The Bonnet transformation naturally also has mathematical applications. Schoen discovered the gyroid by applying the Bonnet transformation to the surface system formed by the adjoint P and D surfaces (4), and by monitoring the movements of the axes of rotation in the I-WP and H surfaces when subjected to isometric bending, we have been able to survey these systems and to show that the I-WP system contains no new embedded surfaces, but that it is related to itself via the Bonnet transformation and a solid rotation (5, config 3) and that the H surface family does contain a hitherto unknown periodic member without self intersections for a specific value of the c/a ratio.

Fig 3. The Bonnet transformation of the I-WP surface at 0°, 20°, 40° and 60°. Note how the surface closes up at 60°.
In figure 4 the Bonnet family of the H surface is shown along the three fold axis. It is clear that only two \( \Theta \) values between 0 and 90 degrees may yield periodic surfaces, and by turning the image 90 degrees, it can clearly be seen that the surface contains gaps, due to a c/a ratio that is not compatible with an embedded surface. By identifying the points indicated by the arrows a set of equations is arrived at, and by numerically solving these, the correct c/a ratio is determined.

Fig 4. The Bonnet transformation of the H surface. View along the three fold axis.

a)0\(^\circ\), b)20\(^\circ\) c)35\(^\circ\) d)50\(^\circ\) e)64.21\(^\circ\) f)75\(^\circ\) g)90\(^\circ\)
Fig 5. The Bonnet transformation of the H-surface viewed along a two fold axis. The Bonnet angle is 64.2° and fig a and b show the difference between an untuned (a) and a tuned (b) c/a ratio.

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