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H. HOPF'S QUADRATIC DIFFERENTIAL AND A WEIERSTRASS FORMULA FOR GENERAL SURFACES AND SURFACES OF CONSTANT MEAN CURVATURE

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Abstract — Using H. Hopf’s quadratic differential \( \Phi \, dw^2 \) we find a representation formula for general surfaces in \( \mathbb{R}^3 \) similar to the classical Weierstrass formula for minimal surfaces. For surfaces of constant mean curvature with prescribed \( \Phi \, dw^2 \) an integrability condition with the properties of a conservation equation is derived. We continue the work of K. Kenmotsu on surfaces of prescribed mean curvature.

1 — THE CLASSICAL WEIERSTRASS FORMULA FOR MINIMAL SURFACES IN \( \mathbb{R}^3 \)

In the theory of minimal surfaces we have the well known Weierstrass formula:

\[
(x_1, x_2, x_3) = \text{Re} \int \left( \frac{1}{2} (1 - g^t), \frac{1}{2} (1 + g^t), g \right) \, fdw,
\]

where \( w = u + iv \) are isothermal coordinates in a domain \( D \), \( g \) is the Gauss function and \( fdw \) is an ordinary differential on \( D \). If the complex analytic function \( w = w(z) \) describes a parameter transformation, then \( f(w) \frac{dw}{dz} = \mathcal{P}(z) \).

2 — H. HOPF’S QUADRATIC DIFFERENTIAL \( \Phi \, dw^2 \) FOR GENERAL SURFACES

Let \( S \) be a regular surface in \( \mathbb{R}^3 \) defined by

\[
X(w): D \to \mathbb{R}^3, \quad X(w) = (x_1(u,v), x_2(u,v), x_3(u,v))
\]

with isothermal coordinates \( w = u + iv \). Then the first fundamental form is

\[
ds^2 = E(du^2 + dv^2) = E|dw|^2.
\]

Let \( L, M \) and \( N \) be the coefficients of the second fundamental form and

\[
\Phi := \frac{1}{2} (L-N) - iM
\]

the complex valued function introduced by H. Hopf, see \cite{1}, p. 136-8. This function is useful in surface theory, for example the Codazzi equations can be shortly written

\[
\Phi(w) = EH_{\bar{w}},
\]

where \( H \) is the mean curvature and \( \frac{\partial \Phi}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right), \frac{\partial \Phi}{\partial \bar{w}} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \). If the complex analytic function \( w = w(z) \) describes a parameter transformation, then

\[
\Phi(w) \left( \frac{dw^2}{dz} \right) = \mathcal{P}(z)
\]
and $\Phi$ transforms like a quadratic differential. Since
\[ |\Phi| = E \frac{1}{H^2-K} = E \frac{|k_2-k_1|}{2} \]
with the Gauss curvature $K$ and the principal curvatures $k_\nu$, the umbilic and flat points of $S$ are the zeros of $\Phi$.

For surfaces of constant mean curvature or minimal surfaces
\[ \Phi W = 0, \]
$\Phi$ is complex analytic in $w$ and $\Phi dw^2$ is an analytic quadratic differential in the sense of the classical theory of Riemann surfaces.

Remarks: If $S$ is a surface of class $C^3$ isothermal coordinates can be constructed, see [1], p. 99. In the following existence and continuity of derivatives will be assumed without explicit mention. Surfaces of constant mean curvature and minimal surfaces are real analytic.

### 3 - $g$, $fdw$ AND $\Phi dw^2$ IN THE THEORY OF MINIMAL-surfaces

These three objects are not independent, it is
\[ \Phi = -fg'. \]
Proof: Start with $\Phi = -2X^i Y_i$ in /1/, p. 138. The normal vector is
\[ Y = \frac{1}{1+|g|^2} (2 \Re g, 2 \Im g, |g|^2-1), \]
for minimal surfaces we have
\[ X^i_w = \frac{1}{2} (X^i - iX^j) = \frac{1}{4} f ((1-g^2), i(1+g^2), 2g), \]
and with $X_u \cdot Y = X_{iv} Y = 0$ and $g = a + ib$ we get the scalar product
\[ \Phi = -2 \left[ \frac{1}{4} f ((1-g^2), i(1+g^2), 2g) \right] \left[ \frac{1}{1+|g|^2} (a_u - ia_v, b_u - ib_v, a(a_u - ia_v) + b(b_u - ib_v)) \right]. \]
Using $g' = a_u + ib_v$ and the Cauchy-Riemann equations $a_u = b_v, a_v = -b_u$ we come to (9), q.e.d. The product $fg'$ is a quadratic differential.

Replace $fdw$ in (1) by $-\frac{\Phi dw^2}{g^2 dw}$, then
\[ (1') \quad (x_1, x_2, x_3) = -\Re \int \left( \frac{1}{2} (1-g^2), \frac{1}{2} (1+g^2), g \right) \frac{\Phi}{g} \, dw. \]
Surprisingly a similar formula holds for arbitrary surfaces $S$.

### 4 - $\Phi dw^2$ AND A WEIERSTRASS FORMULA FOR GENERAL SURFACES

Let $S$ be a regular surface defined by (2) with isothermal coordinates. The normal vector $Y = (n_1, n_2, n_3)$ describes the Gauss map and by stereographic projection we get the complex valued Gauss function $g = \frac{n_1 + i n_2}{1 - n_3}$.

Theorem 1: Let $S$ be a regular surface in $\mathbb{R}^3$ defined by (2) with isothermal coordinates $w = u + iv$. Let $g$ be the Gauss function and $\Phi dw^2$ Hopf's quadratic differential. Then following representation formula holds:
\[ (10) \quad (x_1, x_2, x_3) = -\Re \int \left( \frac{1}{2} (1-g^2), \frac{1}{2} (1+g^2), g \right) \frac{\Phi}{g} \, dw. \]
For minimal surfaces $g$ is meromorphic, $g_w = g'$, and we come back to (1').
Proof: The following equalities are due to Kenmotsu /2/:
\[(11) \quad \frac{\partial}{\partial w} (x_1, x_2, x_3) = - \frac{1}{H} \frac{g_w}{(1+|g|^2)^2} \left(1, -1, (1+g^2), 2g\right),\]
\[(12) \quad \Phi g_w = E H g_w',\]
\[(13) \quad E = 4 \frac{|g_w|^2}{(1+|g|^2)^2} = \frac{1}{4} \frac{(1+|g|^2)^2}{|g_w|^2}.\]

Attention: K. Kenmotsu uses a \(\Phi\)-function different from Hopf's \(\Phi\)-function. If we denote Kenmotsu's function with \(\Phi_K\), the relation \(\Phi_K = \Phi/E\) holds. We have introduced Hopf's function above.

Using (12), (13) and \(\sqrt{g_w} = \bar{g}_w\) we find
\[(14) \quad g_w^{3/2} = \frac{1}{4} H \Phi (1+|g|^2)^2,\]
which will be important in the following.

Now we can prove (10). Using (11) and (14) we get
\[
\frac{\partial x_1}{\partial w} = - \frac{1}{H} \frac{g_w}{(1+|g|^2)^2} (1-g^2) = - \frac{1}{4} \frac{\Phi}{g_w} (1-g^2),
\]
\[- \text{Re} \int \frac{1}{2} (1-g^2) - \frac{\Phi}{g_w} \text{d}w = 2 \text{Re} \int \frac{\partial x_1}{\partial w} \text{d}w = \int \left(\frac{\partial x_4}{\partial u} + \frac{\partial x_4}{\partial v} \text{d}v\right) = x_1,
\]
and the same for \(x_2\) and \(x_3\), q.e.d.

Remarks: According to (13) \(g_w = 0\) iff \(\Phi = 0\) and \(\Phi/g_w\) is not singular at points and lines where \(g_w = 0\). If \(g_w = 0\) in \(D\), then \(\Phi = 0\), all points are umbilic points or flat points and \(S\) lies on a sphere or in a plane. These two special cases must be excluded in theorem 1.

5 — THE FORMULA OF KENMOTSU

Using (11) K. Kenmotsu proved (see /2/, (4.4))
\[(15) \quad \left(x_1, x_2, x_3\right) = - \text{Re} \int \left((1-g^2), 1(1+g^2), 2g\right) \frac{2g_w}{H(1+|g|^2)^2} \text{d}w.
\]

6 — COMPARISON OF (10) AND (15)

With his important formula (15) K. Kenmotsu can construct surfaces of prescribed mean curvature. By introducing Hopf's differential \(\Phi\) \(\text{d}w^2\) into (15) we come to formula (10), which is closer to the formula of Weierstrass. See also /2/, (3.6).

If \(g_w \equiv 0\), then \(\Phi \equiv 0\) and \(S\) lies on a sphere. The plane case will be excluded now. We cannot use (10) but (15). Problem 1: Set \(g = \bar{w}\) and \(H = 1\) in (15) and construct the unit sphere.

If \(S\) is a minimal surface (\(H = 0\)) we cannot use (15) but (10). If you want to construct minimal surfaces and their constant mean curvature companions (10) is preferable.

7 — INTEGRABILITY CONDITIONS FOR SURFACES OF CONSTANT MEAN CURVATURE

In the preceding sections we started with surfaces \(S\) and gave representation formulas to reconstruct \(S\). For applications it is more interesting to start with \(H\) or \(\Phi\) and \(g\) and to construct new surfaces \(S\). Now we treat the constant mean curvature case.
Let $S_\Phi$ be a surface of constant mean curvature defined by (2) with isothermal coordinates $w = u+iv$, $\Phi$ the Gauss function and $\Phi \, dw^2$ Hopf's analytic quadratic differential. The integrability condition of Kenmotsu is

\begin{equation}
\frac{g_{ww}}{} - \frac{2g_{w}}{1+g^2} g_{ww} = 0,
\end{equation}

see /2/, (4.3). This means: The Gauss function $g$ of $S_\Phi$ fulfills (16), and contrary, if $g$ is a solution of (16) with $g_{ww} \neq 0$, then you can construct a branched surface $S_\Phi$ using (15), see /2/, theorem 4. In /3/, p. 40, there are further remarks on branch points.

To construct surfaces $S_\Phi$ of prescribed $\Phi \, dw^2$ we prove

\section*{Theorem 2: Given $S_\Phi$, $g$ and $\Phi \, dw^2$ as above. If $w \in D$ is a point with Gauss curvature $K(w) \neq 0$, then (14) and (16) are equivalent.}

Proof: Since $\Phi_w = 0$, (14) is equivalent to $\frac{\partial}{\partial w} \left[ \frac{4g_{ww}}{1+g^2} \right] = 0$ or

\begin{equation}
g_{ww}g_{w} + g_{ww}g_{w} - \frac{2}{1+g^2} g_{ww}(g_{ww}+g_{w}g_{w}) = 0.
\end{equation}

If $A+iB = 0$ is an abbreviation for (16) and $g = a+ib$, then

\begin{equation}
(A+iB)g_{w} + (A-iB)g_{w} = 0 \quad \text{and} \quad (Aa_{u}+Ba_{v}) - i(Aa_{v}+Ba_{u}) = 0
\end{equation}

are abbreviations for (17). From the first part in (17') we see that (17) or (14) is a consequence of (16), from the second part we see that (16) is a consequence of (17), if the Jacobian determinant of the g-mapping

\begin{equation}
J = a_{u}b_{v} - a_{v}b_{u} \neq 0.
\end{equation}

Since $J$ measures the area distortion of the g-mapping, $J = 0$ iff $K = 0$, q.e.d.

Remarks: To find the Gauss function $g$ of a surface $S_\Phi$ we have to solve the harmonic equation (16) or the nonlinear first order PDE (14) with complex analytic $\Phi$. (14) may be considered as a first integral (conservation equation) belonging to (16). Compare with the oscillator and his equations $x + \phi x = 0$ and $\dot{x}^2 + \phi^2 x^2 = 2E$. Instead of the energy $E$ we take $H \Phi$. — In /2/, (4.1), we find the integrability condition of Kenmotsu for general surfaces. D. Hoffman and R. Osserman /3/ gave a third order PDE as integrability condition, which surprisingly only depends on $g$. Problem 2: Derive a conservation equation for the general case.

\section*{8 — CONSTRUCTION OF SURFACES OF CONSTANT MEAN CURVATURE}

Since surfaces $S_\Phi$ with vanishing $K$ lie on right circular cylinders, we have

\section*{Theorem 3: Take any complex domain $D$ or any Riemann surface and an analytic quadratic differential $\Phi \, dw^2 \neq 0$ on $D$. Construct a solution $g$ with $J \neq 0$ of the integrability condition (14) with constant $H$. Then the representation formula (10) gives a branched surface $S_\Phi$ of constant mean curvature $H$. On the contrary, every surface $S_\Phi$ not lying on a sphere or a right circular cylinder can be constructed on this way. The isolated zeros of $\Phi$ are umbilic points of $S_\Phi$.}

Problem 3: Determine $g$ and $\Phi$ of the right circular cylinder and use (10).

\section*{9 — CONSTRUCTION OF CONSTANT MEAN CURVATURE COMPANIONS OF MINIMAL SURFACES}

Let $S_\Phi$ be a minimal surface with $g$ and $\Phi \, dw^2$ (instead of $f \, dw$). To construct constant mean curvature companions we take the same $D$ and $\Phi \, dw^2$ and a new Gauss function $G(w) = g(w) + \varepsilon(w)$. $G$ and $\varepsilon$ are not complex analytic. Since $g_{w} = 0$ and $g_{w} = g$, the integrability condition (14) is

\begin{equation}
(g' - \varepsilon'_w)\varepsilon_w = \frac{1}{4} H \Phi (1+|g+\varepsilon|^2)^2.
\end{equation}
Theorem 4: If $E_\varepsilon$ is a one parameter family of solutions of (19) with $E_\varepsilon \to 0$ for $H \to 0$ in $D$, then $D, g \in E_\varepsilon, \phi \, dw^2$ and the representation formula (10) give a one parameter family of companions of $S_\varepsilon$. Since there is the same $\phi \, dw^2$, the umbilic points of the companions come from the flat points of $S_\varepsilon$.

Problem 4: If a parabola rolls on a straight line, the focus describes a catenary, the meridian of the catenoid. With a hyperboloid you can get on this way the hyperbolic surfaces of Delaunay of constant mean curvature. In this Delaunay class we may find companions of (a part of) the catenoid. Determine the $\varepsilon$-functions.

Approximation: Linearising (19) we get
\begin{equation}
\varepsilon \frac{\Phi}{H^2} \left(1 + |g|^2 \right)^2 = -\frac{1}{4} H f \left(1 + |g|^2 \right)^2
\end{equation}
with $f = -\frac{\Phi}{H^2}$. The inhomogeneous Cauchy-Riemann equation (20) has the solution
\begin{equation}
\varepsilon H(w) = \frac{H}{4B} \iint_D \frac{f(z)(1 + |g(z)|^2)^2}{z-w} \, dxdy, \quad z = x+iy.
\end{equation}
Now theorem 4 can be used.

Problem 5: D.M. Anderson has constructed constant mean curvature companions of the periodic minimal surfaces of H.A. Schwarz, see /4/, p. 240. Use the method above to construct the companions.

10 - GAUSS AND TOTAL CURVATURE

Let $S$ be a regular surface defined by (2) with isothermal coordinates, $g$ the Gauss function and $\phi \, dw^2$ Hopf's differential. From (7) and (14) we get
\begin{equation}
K = 16 \frac{|g_w|^2(|g_w|^2-|g|^2)^2}{|\phi|^2(1+|g|^2)^2}.
\end{equation}
Note: $|g_w|^2-|g|^2 = J$.

The curvatura integra is
\begin{equation}
C = \iint_S Kd\sigma = \iint_D KEdudv = -4 \iint_D \frac{J}{(1+|g|^2)^2} \, dudv.
\end{equation}
We see the area distortion $J$ of the $g$-mapping and the spherical area distortion $4J/(1+|g|^2)^2$ of the Gauss map.

For minimal surfaces $g_w = 0$, $g_w = g'$, $\phi = -fg'$ and we get the well known formula
\begin{equation}
K = -\left[\frac{4|g'|^2}{(1+|g|^2)^2}\right]^2.
\end{equation}

11 - FINAL REMARKS

Complete abelian minimal surfaces in $\mathbb{R}^3$ can be constructed with compact Riemann surfaces $D$. Two subclasses are the complete minimal surfaces of finite total curvature $C$ and the periodic surfaces, which are of interest in this workshop. A geometric characterization is given in /5/.

To every compact Riemann surface $D$ you can construct complete minimal surfaces of finite $C$, see /6/. Only to special Riemann surfaces $D$ you can construct periodic minimal surfaces without boundary components. For example the genus $p$ of $D$ must be at least 3, since there are three independent differentials of first kind, $f dw$, $gf dw$ and $g^2 f dw$, see /7/, p. 35. The most simple Riemann surfaces of genus $p = 3$ are the hyperelliptic surfaces generated by \{(w-a_1)\ldots (w-a_8)\}. So it is not surprising that the first Schwarz surfaces are constructed with \{(w-14w^2+8w^3)\}. \[C7-167\]
Since $g$, $fdw$ and $\Phi dw^2$ are analytic functions and differentials on compact Riemann surfaces $\mathbb{D}$, we have the formulas of Riemann-Hurwitz and Riemann of the classical function theory, for example

\[(25) \quad \text{number of zeros of } \Phi dw^2 - \text{number of poles} = 4p - 4.\]

If we translate these formulas into geometric language, we find the Gauss-Bonnet theorem and a flat point formula. For abelian minimal surfaces this was done in [5]. For periodic minimal surfaces without boundary components we deduce the well known formula $C = 2\pi \chi$ for the total curvature of a fundamental part and

\[(26) \quad \sum_{F} k = 4p - 4\]

for the sum of the flat points, counted with multiplicity, see [5], (5), and [7], p. 33. According to Hopf this can also be deduced from Poincaré's index theorem.

If $S_M$ is a surface of constant mean curvature, then

\[(27) \quad \sum_{U} \kappa = 4p - 4\]

for the sum of the umbilic points in a fundamental part, counted with multiplicity.

The results of this last section I had in mind when I wrote my abstract with the title "Global formulas for compact Riemann surfaces, minimal surfaces and surfaces of constant mean curvature" at the end of July. Then I came to the investigations of the preceeding sections via Hopf's quadratic differential.

REFERENCES


Berlin, im September 1990.