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CRYSTALLOGRAPHIC ASPECTS OF MINIMAL SURFACES

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Abstract - Symmetry properties of 3-periodic minimal surfaces subdividing $\mathbb{R}^3$ into two congruent regions are discussed. The relation between the order of a flat point and its site symmetry is established. Explicit formulae are given for the calculation of the genus of such a surface depending on the kind of surface patches that build up the surface. Making use of 2-fold axes that have to be embedded in a surface with given symmetry new families of minimal balance surfaces have been derived. Corresponding lists are referred to the different kinds of surface patches.

1 - INTRODUCTION

A minimal surface in $\mathbb{R}^3$ is defined as a surface the mean curvature $H$ of which is zero at each of its points:

$$H = \frac{1}{2} (k_1 + k_2) = 0.$$

Thus the two main curvatures $k_1$ and $k_2$ are equal in magnitude but opposite in sign for each point of a minimal surface.

From the crystallographic point of view those minimal surfaces are especially interesting which are periodic in three independent directions and, therefore, show space-group symmetry. Among them mainly those surfaces which are free of self-intersections seem to be of practical significance, e. g. as biological membranes or amphiphilic films.

Each 3-periodic (minimal) surface without self-intersections subdivides $\mathbb{R}^3$ into two disjunct regions. Each such region is connected - but not simply connected - and forms a 3-periodic infinite labyrinth. The two labyrinths interpenetrate each other with the (minimal) surface as their common interface. Both labyrinths may be either congruent or different, examples for both being known for a long time past /1,2,3/. In the former case the (minimal) surface has special symmetry properties which may be used to derive new kinds of such surfaces.
2 - SYMMETRY OF MINIMAL BALANCE SURFACES

An intersection-free 3-periodic (minimal) surface that subdivides $\mathbb{R}^3$ into two congruent labyrinths has been called a (minimal) balance surface [4]. Each balance surface is uniquely related to a group-subgroup pair $G-S$ of space groups with the following properties: $G$ describes the full symmetry of the surface. Then an isometry of $G$ either maps each side of the surface and each labyrinth onto itself or it interchanges the two sides and the two labyrinths. $S$ is a uniquely defined subgroup of index 2 containing all those isometries of $G$ that do not interchange the two sides of the surface and the two labyrinths.

If the two sides of a balance surface are envisioned in different colours it is obvious that certain black-white space groups may also be used to describe the symmetry of such a surface [5].

The following considerations show that not all space-group pairs with index 2 are compatible with balance surfaces: If $G-S$ describes the symmetry of a given balance surface then each symmetry operation $g \in G$ with $g \notin S$ necessarily interchanges the two sides of the surface and, as a consequence, if there exist fixed points of $g$, the surface has to run through all these fixed points. That means, the corresponding symmetry element (rotation axis, mirror plane, rotoinversion point) in total has to be embedded in the surface. Especially, the following rules hold:

1. If $g \in G$, $g \notin S$ is a mirror reflection, the corresponding mirror plane must be contained in any surface with symmetry $G-S$. As an intersection-free 3-periodic surface cannot comprise an entire plane, $G-S$ cannot be the symmetry of a balance surface.

2. If $g \in G$, $g \notin S$ is a 2-fold, 4-fold or 6-fold rotation (a 3-fold rotation cannot occur because of the subgroup index 2), the corresponding rotation axis must be entirely embedded within any surface with symmetry $G-S$. As a consequence, the surface shows self-intersection along this axis in case of a 4-fold or 6-fold rotation and again $G-S$ cannot describe the symmetry of a balance surface. A 2-fold rotation axis, however, does not give rise to self-intersections and may be contained within a balance surface. As already Schwarz [1] had shown, each straight line that is embedded in a minimal surface is a 2-fold rotation axis of this surface.

3. If $g \in G$, $g \notin S$ is a $\bar{1}$-, $\bar{3}$-, or $\bar{4}$-operation, the corresponding rotoinversion point has to lie on each surface with symmetry $G-S$. $\bar{5}$-operations can be excluded because $\bar{5}$ gives a mirror reflection [cf. (1)].

The compatibility with balance surfaces has been studied for all 1156 types[1] of group-subgroup pairs of space groups with index 2. For the reasons described above 509 of these types are incompatible with balance surfaces. The remaining 547 types are distributed on the crystal families (referred to $G$) as follows: 34 cubic, 67 trigonal/hexagonal, 168 tetragonal, 231 orthorhombic, 44 monoclinic and 3 triclinic ones [6].

For 352 of these 547 types $G$ contains 2-fold rotations that do not belong to $S$, for 88 types such 2-fold rotations do not exist but $G$ contains additional $\bar{1}$-, $\bar{3}$-, or $\bar{4}$-operations in comparison with $S$. For the 107 types left there do not exist symmetry operations with fixed points that belong to $G$ but not to $S$.

[1] Two group-subgroup pairs of space groups are assigned to the same type if the two groups as well as the two subgroups are mapped onto each other by conjugation with the same affine mapping. Enantiomorphic pairs are not distinguished.
A minimal surface in \( \mathbb{R}^3 \) fulfills the conditions \( k_1 + k_2 = 0 \) for each of its points \((k_1, k_2)\) main curvatures). For most points this defining condition holds with \( k_1, k_2 \neq 0 \). Then the area around the considered (ordinary) point has a saddle-like shape. For exceptional points, however, \( k_1 = k_2 = 0 \) holds. These points are called the flat points of the surface. As for any point of a minimal surface the relation \( |k_1| = |k_2| \) is fulfilled the set of all flat points coincides with the set of all points with zero Gaussian curvature \( k_1 \cdot k_2 \).

In the surrounding of a flat point a minimal surface shows \( j \geq 2 \) valleys separated by \( j \) ridges. If a tiling on the surface is constructed such that all flat points lie on vertices and the edges are defined by lines of curvature that connect the flat points, then at least six tiles meet at each flat point. The best known example for a flat point is the "monkey saddle" with \( j = 3 \). It has already been observed for the classical 3-periodic minimal surfaces of Schwarz /1/.

For any point of an intersection-free minimal surface the degree of its flatness may be characterized by a non-negative integer \( \beta \), called its order. Let \( P_0 \) be a point of a minimal surface and \( n_0 \) the normal vector at that point. According to Hyde /7/ the order \( \beta \) of \( P_0 \) may be derived as follows: A second point \( P \) is moved on the surface around \( P_0 \) and the direction of its normal vector \( n \) is considered during this motion. If \( P_0 \) is an ordinary point, \( n \) rotates once around \( n_0 \) during one revolution of \( P \) around \( P_0 \). If, however, \( P_0 \) is a flat point, \( n \) rotates \( \beta + 1 \) times around \( n_0 \) per one revolution of \( P \). The order \( \beta \) of \( P_0 \) is then defined as \( \beta = \beta - 1 \).

Accordingly, an ordinary point has order \( \beta = 0 \), whereas the order of a flat point may be any positive integer. For 3-periodic minimal surfaces flat-point orders up to 4 have been observed /8/. The number of valleys or ridges surrounding a flat point of order \( \beta \) is given by \( j = \beta + 2 \). For (flat) points of given order \( \beta = 4 \) the geometrical situation is illustrated in Figs. 1 to 5. The diagrams refer to the maximal site symmetry compatible with the respective order. The left part of each figure shows the surrounding of the (flat) point. The arrows represent projections of the normal vectors of the surface. The right part illustrates in a stereographic projection the change of the direction of the normal vector along the closed path indicated at the left.

The relation between \( \beta \) and the maximal site symmetry can easily be expressed if rotoreflections \( N \) instead of rotoinversions \( N \) are considered: The maximal site symmetry is \( \tilde{N}m2 \) or \( \tilde{N}m \) for \( \beta \) even or \( \beta \) odd, respectively, with \( N = 2j = 2\beta + 4 \).

For 3-periodic minimal surfaces this maximal site symmetry can only be realized for \( \beta = 0 \) or \( \beta = 1 \). Flat points with any higher value of \( \beta \) and maximal site symmetry occur, for example, on a special kind of 1-periodic minimal surfaces, called saddle towers /9/.

![Fig. 1 - Ordinary point with \( \beta = 0 \) and maximal site symmetry \( \tilde{4}m2-2mm \).](image_url)
Fig. 2 - Flat point with order 1 (monkey saddle) and maximal site symmetry $3m\bar{3}m$.

Fig. 3 - Flat point with order 2 and maximal site symmetry $\bar{8}m2-4mm$.

Fig. 4 - Flat point with order 3 and maximal site symmetry $\bar{5}m\bar{5}m$.

The relation between the orders of (flat) points of minimal surfaces and their possible site symmetries are summarized in Fig. 6. In analogy to the symmetry description of minimal balance surfaces with group-subgroup pairs of space groups, group-subgroup pairs of point groups are used, if the site symmetry of a (flat) point contains symmetry operations that interchange the sides of the surface. If, however, all site-symmetry operations preserve the sides of the surface, only one symbol is given. For minimal surfaces that subdivide $\mathbb{R}^3$ into two non-congruent regions only the latter case may occur.
Fig. 5 - Flat point with order 4 and maximal site symmetry $\bar{1}2m2-6mm$.

Fig. 6 - Group-subgroup diagram showing the maximal site symmetry and all possible crystallographic site symmetries for (flat) points with order $\leq 4$.

As Fig. 6 shows, most site symmetries of points on intersection-free minimal surfaces in $\mathbb{R}^3$ necessarily enforce these points to be flat points of some minimal order. On the other hand, only points with site symmetry $4m2-222, 4-2, 222-2, 2mm, 2, m$ or $1$ can be ordinary points on minimal (balance) surfaces.

The flat points of all minimal balance surfaces described so far have been tabulated by Koch & Fischer /8/.

4 - GENERA OF MINIMAL BALANCE SURFACES

A non-periodic surface in $\mathbb{R}^3$ has genus $g$, if it may topologically be deformed into a sphere with $g$ handles. Consequently, the genus of a 3-periodic minimal surface necessarily has to be infinite. Therefore, a modified definition for the genus of such a surface has been introduced /3/ counting only the number of handles per unit cell. In other words, the surface is embedded in a flat 3-torus to get rid of all translations, and then the conventional definition of the genus may be applied. This procedure corresponds to identifying all opposite faces of a primitive unit cell. In case of a minimal balance surface obviously the unit cell used has to refer to the subgroup $S$, if the symmetry of the surface is described by the space-group pair $G-S$. Otherwise the
identification process would not take into account the existence of two
different sides of the surface and of two labyrinths.

The genus of a 3-periodic minimal surface may be calculated in different ways,
two of which will be discussed in the following:

(1) As has been proposed by Schoen /3/ for each of the two labyrinths
associated with an intersection-free 3-periodic minimal surface a labyrinth
graph may be constructed; each labyrinth graph is entirely located within its
labyrinth; each branch of a labyrinth contains an edge of its graph; each
circuit of a labyrinth graph encircles an edge of the other graph. Then, any
of the two labyrinth graphs may be used to represent the surface. As each
circuit of the graph corresponds to a handle of the surface, the number of
circuits per unit cell (with respect to $S$ in case of a minimal balance
surface) or the number of circuits within the finite graph embedded in the 3-
torus has to be counted to get the genus.

In a modification of a procedure proposed by Hyde /7/ a connected subgraph
containing no translationally equivalent vertices may be separated from the
labyrinth graph. Then, the genus may be calculated as
\[ g = q + \frac{r}{2}, \]
where $r$ is the number of edges connecting the finite subgraph to the rest of
the infinite labyrinth graph and $q$ gives the number of edges that must be
omitted to make the subgraph simply connected. As $r$ equals at least 6 the
genus of a 3-periodic surface without self-intersection must be at least 3.

Keeping in mind the embedding in the flat torus, a more crystallographic
formula may be derived. The number of edges of the finite embedded labyrinth
graph may be calculated from
\[ e = \frac{1}{2} \sum_{i} m_i e_i, \]
where $m_i$ is the multiplicity of the $i$th kind of vertices (referred to a
primitive unit cell of $S$), $e_i$ is the number of edges meeting in such a vertex
and $i$ runs over all symmetrically inequivalent kinds of vertices of the
labyrinth graph. Then
\[ g = e - e_s \]
holds, where $e_s$ is the number of edges in a finite, simply connected graph
with the same number of vertices $v$. From
\[ e_s = v - 1 = \sum_{i} m_i \]
it follows that
\[ g = 1 + \sum_{i} m_i \left( \frac{e_i}{2} - 1 \right). \]

(2) To characterize an intersection-free surface in $R^3$, another number -
called Euler characteristic $\chi$ - may also be used. $\chi$ is closely related to the
genus $g$ by
\[ g = 1 - \frac{\chi}{2}. \]

$\chi$ may be derived in a simple way by dividing the surface into tiles.
\[ \chi = f - e + v \]
holds if $f$, $e$ and $v$ are the numbers of faces (tiles), edges and vertices,
respectively, of an arbitrary tiling of the surface. In case of a 3-periodic
surface, the tiling has to be compatible with the translations of the surface
and the faces, edges and vertices are counted within one unit cell.

Another possibility to calculate the genus of a minimal surface makes use of
the flat points of the surface /7,11/:
where $\beta_i$ is the order of the $i$th flat point within a primitive unit cell of $S$. As the flat points are not easily discernible in all cases, however, it seems more appropriate, to facilitate the search for flat points by the knowledge of $g$.

5 - SETS OF 2-FOLD AXES OR STRAIGHT LINES EMBEDDED IN MINIMAL BALANCE SURFACES

As has been mentioned above there exist exactly 352 types of group-subgroup pairs $G-S$ of space groups with index 2 where $G$ contains more 2-fold rotations than $S$. The corresponding 2-fold axes must lie within each (minimal) balance surface with symmetry $G-S$.

The comparison of such group-subgroup pairs $G_1-S_1$ and $G_2-S_2$ shows the following: the set of 2-fold axes belonging to $G_1$ but not to $S_1$ may be identical with the set of 2-fold axes belonging to $G_2$ but not to $S_2$ although the two space-group pairs differ in their types. With respect to black-white space groups this means that two groups of different types may coincide in that subset of their elements that consists of all colour-permuting 2-fold rotations. If such cases are not treated separately one has to distinguish only 52 different sets of 2-fold axes or 52 different configurations of straight lines which can be embedded into minimal balance surfaces.

Very often two group-subgroup pairs giving rise to one and the same line configuration are connected by subgroup relations. (Such relations can be seen in analogy to subgroup relations between black-white space groups.) In other cases, however, such subgroup relations do not exist. The following example illustrates both situations:

The space-group pairs $P6_3/mmc-P6m2$, $P6_2c-P6$ and $P3m1-P3m1(c'=2c)$ coincide in their sets of 2-fold axes belonging to $G_1$ but not to $S_1$. The corresponding line configuration consists of infinitely many parallel plane nets of regular triangles stacked directly upon each other. A subgroup relation of index 2 exists from the pair $P6_3/mmc-P6m2$ to the pair $P6_2c-P6$, i.e. $P6_2c$ is subgroup of $P6_3/mmc$ and $P6$ is subgroup of $P6m2$. In contrast to that no such relation exists with respect to the pair $P3m1-P3m1(c'=2c)$. As a consequence an $H$ surface (cf. 7) with inherent symmetry $P6_3/mmc-P6m2$ is compatible also with $P6_2c-P6$ but not with $P3m1-P3m1(c'=2c)$. This may be visualized as follows: Half of the plane nets of 2-fold axes belonging to $G=P3m1$ are not contained in $S=P3m1(c'=2c)$ and, therefore, must be embedded in each balance surface with that symmetry. The other half of these nets, however, belong also to the subgroup $S=P3m1(c'=2c)$. These 2-fold axes - that cannot be embedded within any balance surface - are not compatible with an $H$ surface.

It has to be noticed that the inherent symmetry of a minimal balance surface not necessarily coincides with the maximal symmetry found for a given line configuration. The triple-catenoid surfaces $MCl$ (cf. 9), for example, refer to the same line configurations as the $H$ surfaces, but the inherent symmetry is only $P6_3/mcm-P62m$. There exists a subgroup relation of index 3 from $P6_3/mmc$ to $P6_3/mcm$ and from $P6m2$ to $P62m$.

The assignment of the 352 types of group-subgroup pairs to the 52 line configurations is given explicitly in an earlier paper /6/. 18 of these line configurations have the property that the straight lines intersect in such a way that the entire set is 3-dimensionally connected. 12 line configurations disintegrate into infinitely many parallel plane nets, 12 line configurations consist of non-intersecting, parallel or skew 2-fold axes and the remaining 10 configurations are combinations of intersecting and non-intersecting straight lines.

For the systematic derivation of new kinds of minimal balance surfaces those 18 and 12 line configurations are especially useful which consist of 3-dimensionally connected 2-fold axes or of parallel plane nets of 2-fold axes, respectively.
Many of the 3-periodic minimal balance surfaces that have been derived earlier contain 2-fold axes which are embedded within the surface and which form a 3-dimensionally connected set /1,2,3/. If such a surface is cut up along all these 2-fold axes, infinitely many finite surface patches result. In all cases known so far these surface patches have the topological shape of a disc with a skew polygon as boundary (cf. Fig. 7). Then it is possible to construct the entire infinite 3-periodic surface from one given surface patch by continuing this surface patch systematically with the aid of the 2-fold rotations.

The knowledge of the 18 different sets of 2-fold axes that are 3-dimensionally connected enables the systematic and complete derivation of all skew polygons which may be disc-like spanned by a patch of a minimal balance surface /4,6/. It has been proved earlier /12/ that any closed circuit can be disc-like spanned by at least one piece of a minimal surface. To avoid self-intersection, however, a skew polygon that gives rise to a minimal balance surface has to fulfill two conditions: (1) all vertex angles have to be as small as possible within the regarded set of 2-fold axis; (2) no other 2-fold axes out of the regarded set may run through the polygon. A polygon that fulfills these conditions is called a generating circuit of a minimal balance surface.

In total there exist 23 kinds of skew polygons that fulfill these conditions. They refer to 16 of the 18 3-dimensionally connected sets of 2-fold axes. A closer look at these polygons reveals, however, that they give rise to only 15 families of minimal balance surfaces. This smaller number results from the fact, that some of the simpler surfaces can be made up from different surface patches. Then the respective smallest surface patch refers to the full symmetry of the surface, and some of these patches may be united to form a larger one if the symmetry is decreased.

If one refers to a tiling into such disc-like surface patches the Euler characteristic $\chi$ of the surface may be calculated as

$$\chi = f \left(1 - \frac{e_p}{2}\right) + \sum_{i} v_i \cdot v_i.$$

$f$ means the number of surface patches per primitive unit cell of $S$, and $e_p$ the number of edges or vertices of one of these surface patches or of one generating circuit. $v_i$ is the multiplicity of the $i$th kind of symmetrically equivalent vertices within the tiling (referred to the same unit cell).
Tab. 1 - Minimal balance surfaces built up from disc-like surface patches.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Genus</th>
<th>Generating circuit</th>
<th>Space-group pair</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>3</td>
<td>4-gon</td>
<td>Im3m-Pm3m</td>
<td>/1/</td>
</tr>
<tr>
<td>C(P)</td>
<td>9</td>
<td>8-gon</td>
<td></td>
<td>/2/</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
<td>4-gon</td>
<td>Pn3m-Fd3m</td>
<td>/1/</td>
</tr>
<tr>
<td>C(D)</td>
<td>19</td>
<td>12-gon</td>
<td></td>
<td>/3/</td>
</tr>
<tr>
<td>S</td>
<td>11</td>
<td>12-gon</td>
<td>Ia3d-I43d</td>
<td>/4/</td>
</tr>
<tr>
<td>C(S)</td>
<td>9</td>
<td>8-gon</td>
<td>Ia3d-Ia3</td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>9</td>
<td>6-gon</td>
<td>I41 32-P41 3 32</td>
<td>/4/</td>
</tr>
<tr>
<td>C(Y)</td>
<td>13</td>
<td>9-gon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS1</td>
<td>7</td>
<td>8-gon</td>
<td>P62 4 22-P61 5 22(2c)</td>
<td>/6/</td>
</tr>
<tr>
<td>HS2</td>
<td>4</td>
<td>8-gon</td>
<td>P62 4 22-P31 1 12</td>
<td>/6/</td>
</tr>
<tr>
<td>CLP</td>
<td>3</td>
<td>6-gon</td>
<td>P42/mcm-P42/mmc(v)</td>
<td>/1/</td>
</tr>
<tr>
<td>tD</td>
<td>3</td>
<td>5-gon</td>
<td>P42/nmm-I41/amd</td>
<td>/1/</td>
</tr>
<tr>
<td>oCLP</td>
<td>3</td>
<td>6-gon</td>
<td>Pccm-Ccm</td>
<td>/6/</td>
</tr>
<tr>
<td>oDa</td>
<td>3</td>
<td>6-gon</td>
<td>Pnnn-Fddd</td>
<td>/3/</td>
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<tr>
<td>oDb</td>
<td>3</td>
<td>8-gon</td>
<td>Cmam-Imma</td>
<td>/6/</td>
</tr>
</tbody>
</table>

Table 1 gives a survey on these 15 families. The minimal balance surfaces are designated as in earlier papers /4,6/. For each family the genus and the size of the smallest generating circuit are listed, the symmetry is described by a space-group pair, and a reference with respect to its first description (as far as known to the authors) is given.

7 - CATENOID-LIKE SURFACE PATCHES

Some of the minimal balance surfaces described already by Schwarz /1/ or by Schoen /3/ contain sets of 2-fold axes arranged in congruent parallel plane nets. If such a surface is cut up along these 2-fold axes, it disintegrates into infinitely many finite surface patches that look similar to a catenoid. Such a patch may be topologically deformed into a disc with one hole. Its two boundaries (the generating circuits) are formed by two congruent parallel plane polygons originating from neighbouring nets of 2-fold axes (cf. Fig. 8).

In total there exist 9 sets of 2-fold axes or 9 line configurations /6/ that disintegrate into congruent parallel plane nets. Within 2 of them, however, the nets are stacked such, that the polygon centres of one net lie directly above or below the vertices of the neighbouring nets. Therefore, no catenoid-like surface patches can be spanned between these nets. The other 7 line configurations enable the complete derivation of all minimal balance surfaces made up from catenoid-like surface patches.

In 5 cases the congruent nets are arranged in such a way that all vertices and edges are located directly upon each other. Consequently, the catenoid-like surface patches are bounded by two polygons in parallel orientation. This situation occurs with triangular nets (H surface: 60°, 60°, 60°; R3 surface:
30°, 60°, 90°; R2 surface: 45°, 45°, 90°), with square nets (tP surface) and with rectangular nets (oP surface).

In two cases the congruent nets are arranged such that neighbouring nets differ in their orientation. Then the catenoids are bounded by two congruent polygons in different orientation but with polygon centres located directly upon each other. For triangular nets (60°, 60°, 60°) the resulting surfaces have been designated \( rPD \), for rectangular nets with specialized hexagonal metric HS3 surfaces occur.

Catenoid-like surface patches exist only if the distance between neighbouring nets becomes not to large in comparison with the size of the polygons within the nets. The exact limiting value, however, is not known.

The symbol \( rPD \) has been chosen to express that within this family of minimal balance surfaces two limiting cases exist where the rhombohedral symmetry is enhanced. As may be verified by subgroup degradation such minimal surfaces can be regarded either as rhombohedrally distorted cubic \( P \) surfaces or as rhombohedrally distorted cubic \( D \) surfaces /3/.

Fig. 8 - Catenoid-like surface patch of an HS3 surface.

For a minimal balance surface made up from catenoid-like surface patches these catenoids cannot be used as tiles of the surface for the calculation of its Euler characteristic and its genus. In this case, however, one disc-like tile

Tab. 2 - Minimal balance surfaces built up from catenoid-like surface patches.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Genus</th>
<th>Generating circuits</th>
<th>Space-group pair</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>3</td>
<td>3-gons</td>
<td>( P6_3/mmc-P6m2 )</td>
<td>/1/</td>
</tr>
<tr>
<td>( R3 )</td>
<td>13</td>
<td>3-gons</td>
<td>( P6/mcc-P6/m )</td>
<td>/3/</td>
</tr>
<tr>
<td>( tP )</td>
<td>3</td>
<td>4-gons</td>
<td>( I4/mmm-P4/mmm )</td>
<td>/3/</td>
</tr>
<tr>
<td>( R2 )</td>
<td>9</td>
<td>3-gons</td>
<td>( I4/mcm-P4/mbm )</td>
<td>/3/</td>
</tr>
<tr>
<td>( oPb )</td>
<td>3</td>
<td>4-gons</td>
<td>( F\overline{4}m\overline{m}-Cmmm )</td>
<td>/3/</td>
</tr>
<tr>
<td>HS3</td>
<td>7</td>
<td>4-gons</td>
<td>( P6_2.4\overline{2}2-P6_4.\overline{2}22(2c) )</td>
<td>/6/</td>
</tr>
<tr>
<td>( rPD )</td>
<td>3</td>
<td>3-gons</td>
<td>( R\overline{3}m-R\overline{3}m(2c) )</td>
<td>/1/</td>
</tr>
</tbody>
</table>
may be produced from each catenoid by cutting it up between two vertices. By this, an additional edge per catenoid is generated. If $f_v$, $e_v$ and $v_w$ designate the numbers of faces (polygons), edges and vertices, respectively, counted for all nets of 2-fold axes per unit cell, the equation for the Euler characteristic may be rewritten as

$$\chi = v_w - e_v.$$ 

For this $e$ is replaced by $e_v + f$ and $v$ by $v_w$. Making use of the relation $f_v - e_v + v_w = 0$ the genus may be calculated as

$$g = 1 + \frac{e_v}{2}.$$ 

As each catenoid is bounded by two polygons the number $k$ of catenoids is $k = f_v / 2$. From this it follows

$$g = k + 1,$$

where $k$ is the number of catenoids within a primitive unit cell of $S$.

8 - BRANCHED CATENOIDS

3 out of the 12 sets of 2-fold axes disintegrating into parallel plane nets have the property that nets of different kinds are stacked upon each other alternately. In all 3 cases the two kinds of nets differ in the size of their polygons and, therefore, in the number of polygons per unit cell. As a consequence, though it is possible to span a catenoid-like surface patch between two different polygons from neighbouring nets, the continuation of that catenoid with the aid of the 2-fold rotations necessarily would be self-intersecting. One may construct, however, surface patches of a new kind which may be continued to an infinite minimal surface without self-intersection.

These new surface patches have been called branched catenoids. All catenoid-like surface patches described so far are bounded by two congruent flat polygons, their generating circuits. A branched catenoid, on the other hand, has two different generating circuits. It is bounded by a large convex polygon at one end and by several smaller convex polygons with one common vertex at its other end. These smaller polygons are united to one large concave polygon with one point of self-contact (cf. Fig. 9).

There exist twofold, threefold and fourfold branched catenoids. Twofold branched catenoids refer to alternating square nets of 2-fold axes which differ in the size and in the orientation of their squares. Threefold branched catenoids are spanned between nets of regular triangles with different size and orientation. Fourfold branched catenoids correspond to alternating square and triangular ($45^\circ$, $45^\circ$, $90^\circ$) nets. Each of them is bounded by a square at one end and by four triangles with a common vertex at its other end.

![Fig. 9 - A branched catenoid, the surface patch of a BC2 surface.](image)

In analogy to catenoid-like surface patches branched catenoids, too, exist
only if the distance between neighbouring nets of 2-fold axes does not become too large. The genus of such a minimal balance surface may be calculated as

\[ g = \frac{1+b}{2} k + 1. \]

This formula may be derived in a similar way as the corresponding formula for catenoids. Again \( k \) means the number of surface patches (branched catenoids) per unit cell of \( S \), whereas \( b \) is the number of branches (2, 3 or 4).

Tab. 3 - Minimal balance surfaces built up from branched catenoids.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Genus</th>
<th>Generating circuits</th>
<th>Space-group pair</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>( BC1 )</td>
<td>9</td>
<td>3-gon</td>
<td>( P6_3 )</td>
<td>/13/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9-gon</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( BC2 )</td>
<td>7</td>
<td>4-gon</td>
<td>( P4_1 )</td>
<td>/13/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8-gon</td>
<td>( /nnm )</td>
<td></td>
</tr>
<tr>
<td>( BC3 )</td>
<td>6</td>
<td>4-gon</td>
<td>( I422 )</td>
<td>/13/</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12-gon</td>
<td>( -I4 )</td>
<td></td>
</tr>
</tbody>
</table>

9 - MULTIPLE CATENOIDs

Catenoid-like surface patches of minimal balance surfaces are bounded by two parallel congruent flat and convex polygons, branched catenoids by two different parallel flat polygons, one of which is convex whereas the other one is concave with one point of self-contact. A multiple catenoid may be imagined as resulting from fusion of \( n \) neighbouring catenoids. It is bounded by two congruent flat and concave polygons with one point of self-contact each (cf. Fig. 10).

Such multiple catenoids are compatible with those 5 sets of congruent parallel sets of 2-fold axes stacked directly upon each other. If these nets contain inequivalent kinds of vertices, different types of multiple catenoids may be formed.

Fig. 10 - A double catenoid, the surface patch of an MC5 surface.

In total 8 families of minimal balance surfaces have been derived by the authors /14/ and independently by Karcher /9/. There exist double, triple, quadruple and sextuple catenoids.

Again multiple catenoids exist only if the distance between neighbouring nets of 2-fold axes is small enough. The genus of a minimal balance surface that
consists of multiple catenoids is given by

\[ g = mk + 1, \]

where \( k \) is the number of multiple catenoids per unit cell of \( S \) and \( m \) is the number of catenoids that must be combined to result in the multiple catenoid.

Tab. 4 - Minimal balance surfaces built up from multiple catenoids.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Genus</th>
<th>( m )</th>
<th>Generating circuits</th>
<th>Space-group pair</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC1</td>
<td>7</td>
<td>3</td>
<td>9-gons</td>
<td>( P6_3/mcm-P\overline{6}m )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>MC2</td>
<td>13</td>
<td>2</td>
<td>6-gons</td>
<td>( P6/mcc-P6/m )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>MC3</td>
<td>13</td>
<td>3</td>
<td>9-gons</td>
<td>( P6/mcc-P6/m )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>MC4</td>
<td>13</td>
<td>6</td>
<td>18-gon</td>
<td>( P6/mcc-P6/m )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>MC5</td>
<td>5</td>
<td>2</td>
<td>8-gon</td>
<td>( P4_2/mcm-Cmmm )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>MC6</td>
<td>9</td>
<td>2</td>
<td>6-gon</td>
<td>( I4/mcm-P4/mbm )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>MC7</td>
<td>9</td>
<td>4</td>
<td>12-gon</td>
<td>( P4/mcc-P4/m )</td>
<td>/9,14/</td>
</tr>
<tr>
<td>oMC5</td>
<td>5</td>
<td>2</td>
<td>8-gon</td>
<td>( Pccm-P2/m )</td>
<td>/9,14/</td>
</tr>
</tbody>
</table>

10 - INFINITE STRIPS

In addition to the catenoids, the branched catenoids and the multiple catenoids a fourth kind of simple surface patches can be spanned between parallel flat nets of 2-fold axes, namely infinite strips.

Strip-like surface patches are compatible exclusively with those 6 sets of 2-fold axes which disintegrate into rectangular nets /15/. Each strip is bounded by a first zigzag or meander line formed within one quadrangular net and a second such line from a neighbouring net. With respect to the underlying quadrangular nets, zigzag lines run in diagonal directions whereas meander lines extend parallel to the 2-fold axes forming the nets. The two boundary lines of a strip-like surface patch must run parallel and their middle lines must be located directly above each other (cf. Fig. 11).

As the infinite direction of a strip is distinguished from all other directions parallel to the nets of 2-fold axes, only 2-fold rotation axes perpendicular to the nets are compatible with strip-like surface patches. 3-, 4- or 6-fold rotation axes perpendicular to the nets would necessarily give rise to self-intersection of the resulting infinite surface.

7 different kinds of strip-like surface patches may be constructed. For 5 kinds, however, the shape of the strips is such that additional 2-fold axes occur which are embedded within the strips. As a consequence, the resulting minimal surfaces can be also derived with the aid of disc-like surface patches.

2 families of minimal balance surfaces have been found that can be generated by strip-like surface patches but not by smaller ones bounded by 2-fold axes.

The STI surfaces refer to the same rectangular nets as the HS3 surfaces. Adjacent nets are rotated against each other by 120°, but each pair of adjacent nets coincides in the direction of one of their diagonals. The corresponding zigzag lines in this direction span the strip-like surface patches.

Parallel square nets of different size and orientation (cf. BC2 surfaces) form the sets of 2-fold axes that give rise to the second new family. Zigzag lines
within the wider nets and meander lines within the nets with the smaller squares coincide in their directions, and strips may be spanned between a zigzag line and a meander line from adjacent nets. The generated minimal balance surfaces are designated ST2.

Tab. 5 - Minimal balance surfaces built up from infinite strips.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Genus</th>
<th>Space-group pair</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>ST1</td>
<td>7</td>
<td>P62 \cdot 22-P62 \cdot 22(2c)</td>
<td>/15/</td>
</tr>
<tr>
<td>ST2</td>
<td>7</td>
<td>P42 /nbc-P42 /n</td>
<td>/15/</td>
</tr>
</tbody>
</table>

In contrast to the other minimal balance surfaces generated with the aid of parallel plane nets of 2-fold axes ST1 and ST2 surfaces seem not to be restricted with respect to the distance of neighbouring nets.

11 - CATENOIDS WITH SPOUT-LIKE ATTACHMENTS

Schoen /3/ described a family of minimal balance surfaces with the same symmetry as the H surfaces, designated C(H). He derived a corresponding surface patch by deformation of a suitably chosen patch of a C(P) surface. This derivation emphasizes the mirror planes and the plane lines of curvature of a C(H) surface. It is possible, however, to choose larger surface patches which stress the 2-fold axes that are embedded within the surface.

Then the catenoid-like surface patches of an H surface correspond to catenoids with three spout-like attachments of a C(H) surface. In addition to the two triangles that bound the catenoids of an H surface there exist further boundary lines at the ends of the spout-like attachments that do not refer to 2-fold axes. Spouts of three neighbouring catenoids are united to 3-armed handles. Consequently each catenoid is connected via spouts to six neighbouring catenoids.

In analogy to the relation between the H and the C(H) surfaces it is possible to derive further catenoids with spout-like attachments /16/: catenoids with three spouts may be deduced from the '3-sided' R2 and R3 catenoids, catenoids with four spouts from the 4-sided tP and oP catenoids and catenoids with only two spouts also from the oP catenoids (cf. Fig. 12).

The genus of a minimal balance surface made up from catenoids with spout-like attachments is given by

\[ g = sk + 1, \]

where \( k \) is the number of catenoids per primitive unit cell of \( S \) and \( s \) is the number of spouts per catenoid.

A similar view of the C(H) and tC(P) surfaces is discussed by Karcher /9/. Furthermore he proposes to construct complicated new minimal surfaces from
simpler ones by fitting in additional handles. As corresponding example he
describes a minimal balance surface derived from the \( C(P) \) surface by building
in a handle between each pair of opposing disc-like surface patches. Probably
the same procedure may be used in connection with other minimal balance
surfaces, e. g. the \( C(D) \) and the \( C(Y) \) surface.

\[ \text{12 - FURTHER MINIMAL BALANCE SURFACES} \]

So far only 2 families of minimal balance surfaces have been described that
contain disconnected 2-fold axes. The corresponding sets of 2-fold axes may be
visualized as follows: within a space partition into cubes all faces of the
cubes are bisected by 2-fold axes such that the axes do not intersect. The
proposed surfaces may be built up from disc-like surface patches spanned
between parts of six different 2-fold axes. Obviously, such surface patches
cannot be totally bounded by 2-fold axes, but part of their boundaries is not
distinguished by symmetry. A detailed description of these surfaces,
designated \( Y \) and \( C(Y) \), is given in /4,10/. Their existence, however, could
not been proved until now.

All minimal balance surfaces mentioned so far have the common property that
they divide \( R^2 \) into two labyrinths that are not enantiomorphic but directly
congruent. The only surfaces that give rise to two enantiomorphic labyrinths
are the gyroid surfaces of Schoen /3/, designated \( Y \) surfaces in the
following. Though a gyroid surface contains only 3 and 4 rotoinversion centres
that are embedded in the surface but no 2-fold axes, the existence of such
surfaces is proved because they may be obtained by Bonnet transformations of \( P 
\) or \( D \) surfaces.
Karcher /9/ discusses the derivation of new families of minimal surfaces via metrical deformation of known ones. In this way he derived an orthorhombically deformed $P$ surface, designated $oPa$ in the following, that does not contain any 2-fold axes within the surface, but only inversion centres. Nevertheless, the two corresponding labyrinths are not enantiomorphic because the symmetry of such a surface contains mirror reflections.

From the symmetry point of view it may be said that all cubic minimal surfaces may be rhombohedrally deformed, and those belonging to crystal classes $m\bar{3}m$, $432$ or $4\bar{3}m$ may be tetragonally distorted, and that all tetragonal minimal surfaces - not belonging to crystal class $4$, $4$ or $4/m$ - may be orthorhombically deformed in two different ways. Whether or not such a deformation is possible under preservation of the special curvature properties of minimal surfaces cannot be answered with the aid of symmetry.

For the cubic $P$ surfaces and $D$ surfaces all rhombohedral, tetragonal and orthorhombic deformations are known:

<table>
<thead>
<tr>
<th>Family</th>
<th>Symbol</th>
<th>Genus</th>
<th>Space-group pairs</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubic</td>
<td>$P$</td>
<td>1</td>
<td>$I\bar{m}\bar{3}m-Pm\bar{3}m$</td>
<td></td>
</tr>
<tr>
<td>rhombohedral</td>
<td>$rPD$</td>
<td>2</td>
<td>$R\bar{3}m-R\bar{3}m(2c)$</td>
<td></td>
</tr>
<tr>
<td>tetragonal</td>
<td>$tP$</td>
<td>3</td>
<td>$I4/mmm-P4/mmm$</td>
<td></td>
</tr>
<tr>
<td>orthorhombic</td>
<td>$oPa$</td>
<td>3</td>
<td>$I\bar{m}m-Pnmn$</td>
<td>/9/</td>
</tr>
<tr>
<td></td>
<td>$oPb$</td>
<td>3</td>
<td>$Pmmn-Cmmm$</td>
<td></td>
</tr>
</tbody>
</table>

The two orthorhombic kinds of deformation are distinguished by the shape of the orthorhombic unit cells that correspond to the original cube. In the first case the unit cell has 3 right angles and 3 different edge lengths, in the second case it is a prism with a rhombic cross-section.

Tab. 7 - Some further minimal balance surfaces

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Genus</th>
<th>Space-group pairs</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(Y)$</td>
<td>13</td>
<td>$Ia\bar{3}-Pa\bar{3}$</td>
<td>/4,10/</td>
</tr>
<tr>
<td>$\bar{Y}$</td>
<td>21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Y^*$</td>
<td>3</td>
<td>$I\bar{a}3d-I4,32$</td>
<td>/3/</td>
</tr>
<tr>
<td>$oPa$</td>
<td>3</td>
<td>$Immm-Pmmm$</td>
<td>/9/</td>
</tr>
</tbody>
</table>

The symmetry of such metrically distorted surfaces may be derived by subgroup degradation applied to space-group pairs or to black-white space groups /17/.

For the gyroid surfaces a rhombohedral deformation gives rise to symmetry $R\bar{3}c-R\bar{3}2$, a tetragonal distortion results in $I4,22-P4_12_2$, the two kinds of orthorhombic distortion refer to $I2_1,2_1,2_1-P2_12_12_1$ and to $P222-C222$, respectively.

Tetragonal deformations cannot exist for the $Y$ surfaces and the $C(Y)$ surfaces, because their symmetry $Ia\bar{3}-Pa\bar{3}$ does not allow tetragonal subgroups. It is very probable, however, that orthorhombically deformed $Y$ surfaces and $C(Y)$ surfaces with symmetry $Ibc-a-Pbca$ do exist because the sets of 2-fold axes from $Ia\bar{3}$ and $Ibc-a$ correspond to each other.

REFERENCES

