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TWO GENERALISED THEORIES DESCRIBING LIGHT DIFFRACTION BY ULTRASONIC PULSES :
A. GENERATING-FUNCTION METHOD
B. LAPLACE-TRANSFORM METHOD

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RESUME : Commencant par les équations de Maxwell, nous dérivons deux modèles théoriques qui décrivent la diffraction de la lumière par des pulsations ultrasoniques. Dans le premier modèle, basé sur une fonction génératrice, nous trouvons des solutions analytiques pour les amplitudes et les intensités des ondes diffractées, qui sont seulement valable dans le régime de Raman-Nath. Le second modèle est décrit par une méthode numérique, utilisant les transformations de Laplace. Ce modèle décrit aussi bien le régime de Raman-Nath que le régime de Bragg. Les deux méthodes sont comparées avec une théorie existante basée sur l'intégral de diffraction. Nous concluons que nous retrouvons ces derniers résultats considérant le régime extrême de Raman-Nath $\rho = 0$ dans nos modèles. Une règle générale concernant la propriété de l'(a)symétrie du spectre, en incidence normale, est dérivée des équations générales de Raman-Nath, utilisant la méthode de la fonction génératrice.

ABSTRACT : As an extension from previous studies concerning the interaction of light with ultrasound of a purely sinusoidal form, two theoretical models describing light diffraction by ultrasonic pulses are derived starting from the Maxwell equations. Based on a generating function, the first model provides finite solutions for the amplitudes and intensities of the diffraction pattern and is restricted in validity to the Raman-Nath region. The second model consists of a numerical method developed by means of Laplacetransformations and covers as well the Raman-Nath as the Bragg region. Both new methods are compared with a theory based on the diffraction-integral theory, leading to the conclusion that the results of this latter method can be obtained from our models considering the extreme Raman-Nath regime $\rho = 0$. A general rule concerning the (a)symmetry properties of the diffraction spectrum at normal incidence has been derived from the extended Raman-Nath equations, using the generating function method.

1. INTRODUCTION :

Recent developments in Non-Destructive Testing and medical diagnosis encouraged more and more physicists to study pulsed ultrasound instead of continuous plane waves. Our current interest is to describe the interaction of plane wave laser light with ultrasonic pulses at normal incidence. Extending known theories for continuous supersonic waves [1,2] to pulses, we calculate the amplitudes and intensities of the diffracted light in the fraunhofer pattern in two different manners. Both models start from the Maxwell equations and follow the light through a medium with a refractive index which is modulated due to the ultrasonic pressure of the pulse. The time-history of this refractive index can be expressed by a fourier sine expansion which proves that a pulse can be identified with a superposed wave :

$$\mu(x,t) = \mu_0 + \mu \sum_{j=1}^{\infty} \alpha_j \sin(2\pi j [\nu_p^* t - \frac{x}{\lambda_p^*}] + \delta_j) \quad (1)$$

where ν_p^* = repetition frequency

λ_p^* = length of the pulse in the medium

μ_0 = refractive index of the undisturbed medium

μ = maximum variation of the refractive index

$\mu \cdot \alpha_j$ and δ_j are the amplitude and phase of the j-th Fourier-component of the pulse.

Considering Maxwell's equations as a starting point, Leroy [3] derived that the amplitudes of the diffraction orders in the far-field satisfy the extended Raman-Nath equations :

$$2 \frac{d\tilde{\Psi}_n(\zeta)}{d\zeta} - \sum_{j=1}^{\infty} \alpha_j [\tilde{\Psi}_{n-j}(\zeta) \exp(-i\delta_j) - \tilde{\Psi}_{n+j}(\zeta) \exp(i\delta_j)] = i\rho^2 \tilde{\Psi}_n(\zeta) \quad (2a)$$

$$\text{with boundary conditions } \tilde{\Psi}_n(\zeta=0) = \delta_{n,0} \quad (n: -\infty \dots +\infty) \quad (2b)$$

$$\text{where } \zeta = \frac{2\pi}{\lambda} \mu z \quad \text{and} \quad \rho = \frac{\lambda^2}{\mu_0 \mu \lambda_p^{*2}}$$

The solution $\tilde{\Psi}_n(\zeta)$ is the amplitude of the diffracted lightbeam whose direction with regard to the incident laser-beam is defined by the angle $\theta_n = -\arcsin(\frac{n\lambda}{\lambda_p^*})$ while its frequency will be $\nu - n \nu_p^*$ and the intensity is given by

$$I_n(\zeta) = \tilde{\Psi}_n(\zeta) \cdot \tilde{\Psi}_n^*(\zeta) \quad (3)$$

We now propose two methods to integrate the system of difference-differential equations (2).

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2. CALCULATION OF THE FARFIELD AMPLITUDES OF THE DIFFRACTION PATTERN BY MEANS OF A GENERATING FUNCTION METHOD.

Considering $\Psi_n(\zeta)$ to be the coefficients of the Laurent expansion of an unknown generating function $G(\zeta, \eta)$, which we suppose to be holomorphic in an annular region with centre O in the complex η -plane ($G(\zeta, \eta) = \sum_{n=-\infty}^{\infty} \Psi_n(\zeta) \eta^n$ (4)) we find that

$G(\zeta, \eta)$ satisfies the following equation

$$2 \frac{\partial G}{\partial \zeta}(\zeta, \eta) - \sum_{j=1}^{\infty} \alpha_j \left[\eta^j \exp(-i\delta_j) - \eta^{-j} \exp(i\delta_j) \right] G(\zeta, \eta) = i\rho \left[\eta^2 \frac{\partial^2 G}{\partial \eta^2}(\zeta, \eta) + \eta \frac{\partial G}{\partial \eta}(\zeta, \eta) \right] \quad (5a)$$

with boundary conditions $G(\zeta=0, \eta) = 1$

(5b)

Once the function $G(\zeta, \eta)$ is known, the amplitudes of the diffracted light waves may be found by expanding $G(\zeta, \eta)$ into a Laurent-series.

In order to integrate (5) we propose a solution in the form of a series in ρ : $G(\zeta, \eta) = \sum_{k=0}^{\infty} (i\rho)^k G_k(\zeta, \eta)$ (6)

Substituting this expansion into (5) and we obtain a partial differential equation for each integer value of $n \geq 0$ with the appropriate boundary condition. If we assume that ρ is only a small value, we neglect in (6) terms of ρ^2 and higher. By solving these equations for $n = 0$ to 2, we have found an approximate solution for $G(\zeta, \eta)$:

$$G(\zeta, \eta) = G_0(\zeta, \eta) + i\rho G_1(\zeta, \eta) - \rho^2 G_2(\zeta, \eta) \quad (7)$$

Comparing the coefficients of η^n in (4) and (7), we get an analytical expression for the diffracted lightwave of order n in the form of $\Psi_n(\zeta) = \Psi_n^{(0)}(\zeta) + i\rho \Psi_n^{(1)}(\zeta) - \rho^2 \Psi_n^{(2)}(\zeta)$ (8)

An interesting remark however is that $\Psi_n^{(0)}$ corresponds to the amplitude of the n -th diffraction order in the case of extreme Raman-Nath regime ($\rho = 0$) and is exactly the same expression as Neighbors and Mayer [4] found by means of the diffraction-integral theory. As $\Psi_n^{(0)}$ is only the first term in our expression for the diffracted amplitudes, we can consider $i\rho \Psi_n^{(1)}$ and $-\rho^2 \Psi_n^{(2)}$ as corrections to the theory of Neighbors and Mayer (see Fig 1.a and b.).

Fig 1. a. Farfield diffraction pattern of normally incident light on a gaussian shaped ultrasonic pulse (window). Results of the generating function method for $\rho = 0.0$ (—) and $\rho = 9.0E-04$ (Δ) ($v = 2.5$)
b. Farfield diffraction pattern of normally incident light on an exponentially damped ultrasonic pulse (window). Results of the generating function method for $\rho = 0.0$ (—) and $\rho = 1.1E-02$ (Δ) ($v = 2.0$)

For a purely sinusoidal ultrasonic wave, the formulae obtained by Kuliasko et al [1] are exactly reproduced.

3. CALCULATION OF THE FARFIELD AMPLITUDES OF THE DIFFRACTION PATTERN BY MEANS OF A LAPLACE-TRANSFORM METHOD.

Defining $k_1 = v_0^*/v_p^*$ (v_0^* = the centre modulation frequency, v_p^* = the repetition frequency) and $\beta = \arcsin \left[\frac{\lambda}{2\lambda_p^*} \right]$, it is

known from theoretical and experimental studies [4-7] that the primary orders in the fraunhofer diffraction pattern show up in the directions $k_1 \cdot 2\beta$ and at most k_1-1 secondary orders can be observed between two primary orders at every even multiple of the angle β . Also from previous studies we know that, in the case of normal incidence of the laser beam, as many positive as negative primary diffraction orders will contribute to the final spectrum, while the number of secondary orders left and right of a peak can differ for one primary order compared to another. Suppose now that only $2N+1$ primary orders (N positive and N negative) have a non-zero contribution of intensity to the final diffraction spectrum and that for the k -th primary order ($-N \leq k \leq N$) only $P(k)$ secondary orders to the left and $Q(k)$ secondary orders to the right of this peak occur. If we assume that all other amplitudes are vanishing small, the infinite system (2) is reduced to a finite set of equations. Applying the Laplace-transform to the truncated system and substituting

$$\phi_n(s) = \mathcal{L}[\Psi_n(\zeta)] = \int_0^\infty \Psi_n(\zeta) e^{-s\zeta} d\zeta \quad \text{we obtain an algebraic system of equations which can be written in matrix notation as follows:}$$

$$[D - tI] \hat{\Phi}(t) = 2iE \quad (9)$$

where I is the unit-matrix of dimension d with $d = \sum_{j=-N}^N [P(j) + Q(j) + 1]$ and D a hermitian matrix.

$\hat{\Phi}(t) = \Phi(\frac{it}{2})$ with $\Phi(s)$ given by the column matrix :

$${}^t \begin{bmatrix} \Phi_{-Nk_1-P(-N)}(s) \dots \Phi_{-Nk_1+Q(-N)}(s) \Phi_{-(N-1)k_1-P(-N+1)}(s) \dots \Phi_{-(N-1)k_1+Q(-N+1)}(s) \dots \Phi_{-P(0)}(s) \dots \Phi_0(s) \dots \Phi_{+Q(0)}(s) \\ \Phi_{Nk_1-P(N)}(s) \dots \Phi_{Nk_1+Q(N)}(s) \end{bmatrix} \quad \text{and} \quad E = {}^t \begin{bmatrix} 0 \dots 0 & 0 \dots 0 & 0 \dots 1 & 0 \dots 0 \end{bmatrix}$$

The unknown Laplace transformed functions $\hat{\Phi}_n(t)$ can be calculated from (9) using Cramer's method, and applying the Heaviside expansion theorem [8] for the special case in which D has only single eigenvalues we get :

$$\Psi_{kk_1+l_k}(\zeta) = \sum_{p=1}^d \frac{[D - t_p I]_{P,P} \prod_{\substack{m=1 \\ m \neq p}}^d (t_p - t_m)}{\prod_{\substack{m=1 \\ m \neq p}}^d (t_p - t_m)} \frac{it_p \zeta}{2} e^{\frac{it_p \zeta}{2}} \quad (10)$$

with $[D - tI]_{P,P}$ the cofactor of the element on the position $(P, P)_{(k, l_k)}$ in $(D - tI)$, t_p an eigenvalue of D and

$$P = \sum_{j=-N}^j [P(j) + Q(j) + 1] + P(0) + 1 \quad P_{(k, l_k)} = \sum_{j=-N}^{k-1} [P(j) + Q(j) + 1] + l_k + P(k) + 1$$

The intensity of order $-(kk_1 + l_k)$ is then given by the combination of (3) and (10).

Fig 2. : Dependence of the farfield diffraction pattern on ρ . The diffracting pulse is shown in fig. 2a. (from J. Wolf [7])

2b. $\rho = 1.0E-07$ or $\rho_0 = 2.19E-05$

2c. $\rho = 5.0E-03$ or $\rho_0 = 1.09$

2d. $\rho = 1.0E-02$ or $\rho_0 = 2.19$

2e. $\rho = 2.0E-02$ or $\rho_0 = 4.38$

2f. $\rho = 3.0E-02$ or $\rho_0 = 6.57$

2g. $\rho = 7.0E-02$ or $\rho_0 = 15.33$

2h. $\rho = 9.0E-02$ or $\rho_0 = 19.71$

Remark that for this numerical method, no restrictions on the Raman-Nath parameters ρ and v are assumed beforehand, so that, by choosing N and $\{P(k), Q(k)\} \mid k : -N \dots N$ in an appropriate way, this model can cover as well the Raman-Nath as the Bragg region. The influence of ρ on the diffraction pattern caused by interaction of light with the ultrasonic pulse shown in

fig 2a is illustrated in fig 2b-2h. Important changes in the pattern only occur when ρ_0 takes on values higher than 1 ($\rho_0 = \frac{\lambda^2}{\mu_0 \mu \lambda_0^2}$, λ_0^2 the centre modulation wavelength of the pulse) : the number of primary orders diminishes, satellite lines vanish and when ρ is large enough only zeroth, first and minus first primary orders are measurable.

At last, we visualise the convergence of our new Laplacetransformtheory towards the MN-OA-method of E. Blomme and O. Leroy [2] which was derived for continuous waves. Increasing the variance-parameter of a gaussian-shaped pulse, the diffraction pattern changes in this way that satellite lines lose part of their intensity while this lost of energy is compensated by an increase of the intensity of the primary orders (except for the zeroth order). When the variance-parameter is large enough, the farfield spectrum is exactly the same as the one obtained by the MN-OA-method for normal incidence of light on a continuous ultrasonic wave : no satellite lines and a perfect symmetric pattern.

Fig 3. : Convergence of the diffraction patterns of gaussian shaped pulses ($k_1 = 6$) calculated by means of the Laplace-transform method, towards the continuous wave limit (same results as the MN-OA-method [2]). ($v = 2.0$, $\rho = 4.166E-02$ or $\rho_0 = 1.5$)

4. GENERAL THEORY CONCERNING SYMMETRY-PROPERTIES OF THE DIFFRACTION PATTERN

Replacing the complex variable η by $\exp(iA) \eta^{-1}$ (A being any real number) in (4) we find that

$\tilde{G}(\zeta, \eta) = \sum_{n=-\infty}^{\infty} \Psi_n(\zeta) \exp(inA) \eta^n$ satisfies the same partial differential equation and boundary condition as $G(\zeta, \eta)$ provided

$$\text{that : } A = 2\delta_1 + (2k-1)\pi \quad \text{and} \quad \delta_j = j\delta_1 + [(2l-1) - j] \frac{\pi}{2} \quad (k, l \in \mathbb{Z})$$

As the solution is unique, we may conclude that under these restrictions $\Psi_n(\zeta) = \Psi_{-n}(\zeta) \exp(inA)$ so that the diffraction pattern is symmetric with respect to the zeroth order. This condition, which is rather strong, is generally not fulfilled in the case of pulses, so that mostly the diffraction pattern will be asymmetric.

5. CONCLUSION

The two new methods derived from the Maxwell equations demonstrate in a very satisfying way the splitting of the diffraction pattern into primary and secondary orders and the property of asymmetry of the spectrum in the case of normal incidence of plane wave laser light on ultrasonic pulses. Both models gain an advantage in accuracy over the diffraction-integral theory used in previous studies. Especially the Laplace-transform method is a very powerful method which is valid as well in the Raman-Nath as in the Bragg region.

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