AN OPERATOR TOP-DOWN DERIVATION OF "DOUBLY ASYMPTOTIC APPROXIMATIONS"

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ABSTRACT

Some problems involved in fluid-structure interaction in unbounded domains require the computation of the response of an homogeneous acoustic medium to prescribed harmonic motions. The complexity of the submerged structure studied sometimes suggests the use of approximate methods for the numerical anlaysis of this acoustics problem [1]. The present paper intends to propose a derivation of the so-called "Doubly Asymptotic Approximations" (DAA's). This formal top-down derivation, specialized to steady-state motions, relies on an integral representation of the solution of the Helmholtz equation in an unbounded domain. Two asymptotic expansions of this representation are obtained in the low- and the high-frequency ranges, then these expansions are matched. This procedure allows to point out that some geometrical assumptions underlie the validity of high order continuous forms of the DAA's. It suggests further investigations of some interesting open geometry and numerical analysis problems.

0. INTRODUCTION

The Neumann problem for the Helmholtz equation in an unbounded domain will be considered throughout this paper. All unknowns will be assumed to be harmonic in time with a time factor $\exp(+i\omega t)$ which will be omitted in the sequel. Expressed in terms of the fluid displacement potential field φ, the equilibrium equations, the boundary condition and the Sommerfeld radiation conditions read :

$$(P_k)\begin{cases}\Delta\varphi + k^2\varphi = 0 \text{ in } \Omega\\ \partial_n\varphi = g \text{ on } \Gamma\\ \left|\dfrac{\partial\varphi}{\partial r} + ik\varphi\right| = O(\frac{1}{r^2}) \text{ and } |\varphi| = O(\frac{1}{r}) \text{ as } r\to\infty.\end{cases}$$

where k is the wave number ω/c, Ω denotes (Figure 1) the exterior domain and Γ its boundary, n denotes the outer unit normal, ∂_n the normal derivative and the real function g is the prescribed normal displacement field on the surface Γ.

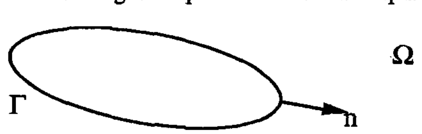

Figure 1

Our attention will be devoted to approximations of the "impedance" operator T which solves the restriction to Γ of the unique solution of (P_k), denoted $\varphi_{|\Gamma}$ (in the following, the subscript $_{|\Gamma}$ will be omitted) :

$$\varphi_{|\Gamma}=T(g).$$

The proposed derivation relies on the Helmholtz integral representation of the solution which reads, for a point x lying on Γ :

$$\frac{1}{2}\varphi(x)= \int_\Gamma [\, \varphi(y) \frac{\partial G_k}{\partial n_y}(x,y) - \frac{\partial\varphi}{\partial n_y} G_k(x,y)\,]\, d\Gamma_y, \quad G_k(x,y) = \frac{\exp(-ikd(x,y))}{4\pi d(x,y)} \tag{1}$$

where d(x,y) denotes the distance between x and y and n_y is the outer normal to the surface Γ at point y. This representation provides the exact impedance, i.e. the exact operator T.

It will be shown that in both low-k and high-k ranges, this representation yields successive approximations of the operator T. Both formal asymptotic expansions will be considered in the two next sections. Their matching will be studied in a third one in order to get continous forms of DAA's. In the last section their discretized counterparts will be considered. This derivation generalizes previously published ones which relied on a modal approach [1] or on a scalar approach for a model problem [2]. It is more detailled in reference [3].

1. LOW-FREQUENCY EXPANSION

When k is assumed to be small compared to 1, asymptotic expansions of the Green function $G_k(x,y)$ and of the restriction to Γ of the solution of (P_k) as

$$\varphi = \sum_i T_i(g), \tag{2}$$

first yields :

$$\varphi_0 = T_0(g) \tag{3}$$

where T_0 solves the restriction to Γ of the solution of the Neumann problem for the Laplace equation.
Then, it may be shown that φ_1 is constant (cf. [4]) and that :

$$\varphi_1(x) = T_1(g) = \frac{i}{4\pi}\int_\Gamma g(y)\, d\Gamma_y.$$

The first order low-frequency approximation of φ therefore reads :

$$\varphi = T_0(g) + kT_1(g) = T_0(g) + \frac{ik}{4\pi}\int_\Gamma g(y)\, d\Gamma_y. \tag{4}$$

2. HIGH FREQUENCY EXPANSION

When k is assumed to be large compared to 1, the formal asymptotic expansion relies on the *limiting absorption principle* [5] and on the method of stationary phase [6]. It is known that the outgoing solution of problem (P_k) may be sought as the limit, when the real positive number ζ tends to 0_+, of the solution φ' of problem (P'_k) :

$$(P'_k)\begin{cases}\Delta\psi + (k-i\zeta)^2\psi = 0 \text{ in } \Omega\\ \partial_n\psi = g \text{ on } \Gamma\\ \int_\Omega |\psi|^2\, d\Omega < \infty\ ,\ \int_\Omega |\nabla\psi|^2\, d\Omega < \infty\end{cases}$$

The Helmholtz integral representation (1) still holds for φ' with the Green function $G_{k'}$, where $k' = k - i\zeta$. It reads as the sum of two integrals (a single layer term S(x) and a double layer term D(x)) over a manifold of $\mathbf{R}^2$.
As k tends to infinity, the main contribution to such integrals is known to result from stationary points of the phase function. As the phase function is here the distance d(x,y) between point x and another point y lying on Γ, we have to consider points y_i where $(x-y_i).n_i = \pm 1$ (Figure 2), that is point x itself and points as y_1, y_2 and y_3.

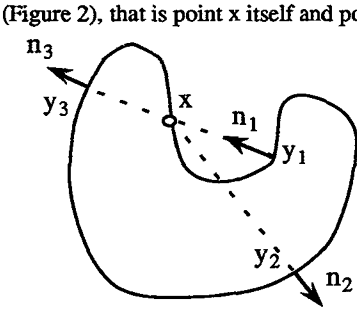

Figure 2

2.1 Contribution of Point x

The singularity of the Green function at point x requires an approximation of the surface Γ in the neighbourhood of point x. Choosing the principal curvature directions as axes in the plane tangent to Γ at point x and the outer normal to Γ at point x as a third axis, any point y of the neighbourhood N(x) may be represented by cylindrical coordinates (ρ,θ).

Developping both integrands in the neighbourhood of x gives the contribution of the point x to the single layer term and the double layer term in (1), that is respectively :

$$S(x) \approx -\frac{g(x)}{2i(k-i\zeta)} + O(\frac{1}{k'^2}), \tag{5}$$

$$D(x) \approx -\frac{\psi(x)\kappa(x)}{2i(k-i\zeta)} + O(\frac{1}{k'^2}). \tag{6}$$

where $\kappa(x)$ is the mean curvature at point x.

2.2 Contribution of Points y_i

For points such as y_i (Figure 2), standard 2-D stationary phase results [5],[6] may be used for the single layer and the double layer terms involved in the Helmholtz integral representation. Both integrals read, as k tends to ∞, to :

$$I(y_i)=\frac{2\pi}{k'}\frac{\exp[-ik'f(y_i)]\exp[i\frac{\pi}{4}\,\mathrm{sign}[(\mathrm{Hess}(f))(y_i)]}{|\mathrm{Hess}(f)(y_i)|}F(y_i) + O(\frac{1}{k'^2}) \tag{7}$$

where Hess(f) and sign[(Hess(f))(y_i)] respectively denote the Hessian of function f and its signature.

The expression of F(y_i) for the double layer integral involves a term which reads :

$$(ik'+\frac{1}{d(x,y_i)})\frac{(y_i-x).n_i}{d(x,y_i)}\psi(y_i).$$

It makes necessary to distinguish points according to whether the outer normal to Γ at point y_i is directed towards point x (Points y_i, i=2,3, on Figure 2 : $\delta(y_i)$ = +1) or not (Point y_1 : $\delta(y_1)$ = -1). Both cases will be considered successively.

2.2.1 Contribution of points where $\delta(y_i)$=+1. For such points, following Bloom [7] who writes the Sommerfeld boundary condition as :

$$|g(y) + (ik+\frac{1}{r})\psi(y)| = O(\frac{1}{r^2}) \text{ as } r \to \infty,$$

we shall admit that (8)

$$|g(y_i) + (ik+\frac{1}{d(x,y_i)})\psi(y_i)| = O(\frac{1}{k^2}) \text{ as } k \to \infty.$$

2.2.2 Contribution of points where $\delta(y_i)$=-1. As far as these points are concerned, from (7), it may be shown that they contribute to the asymptotic expansion of ψ with terms of the order k^{-1}.

Thus the (g, ψ) relationship is not local if Ω is not convex. The assumption that Ω is convex will be made in the sequel and it is necessary for the following derivation to hold.

2.3 Expansion

With this geometrical assumption and for large k, the results of the previous subsections, as ζ tends to 0 are summarized in :

$$g(x) = -ik\varphi(x) - \varphi(x)\kappa(x),$$

or

$$g = -ikI(\varphi) - H(\varphi), \tag{9}$$

where I denotes the identity operator for functions defined on the surface Γ and where H associates to such a function φ the fonction H(φ) defined by :

$$(H\varphi)(x)=\kappa(x)\varphi(x).$$

Approximate relation (9) is equivalent to the plane wave and the curved wave approximations.

3 MATCHING

The so-called Doubly Asymptotic Approximations approach, in both low and high frequency ranges, the approximations got in the previous subsections. The matching is obtained by assuming a relation as :

$$(I + kC_1 - ik^2C_2)(\varphi) = (T_0 + kC_2)(g). \quad (10)$$

where C_1 and C_2 are unknown operators. Moreover, the operator C_2 is supposed to be invertible (and this assumption will have to be checked later).

The unknowns operators C_1 and C_2 will be determined from the comparaison of the second order expansions for the low- and high-frequency range expansion of (10), respectively with (4) and (9). They are shown to give :

$$C_1T_0(T_0^{-1} + H) = -(iT_0 + T_1H), \quad (11)$$

$$C_2(T_0^{-1} + H)T_0 = -iT_0^2 + T_1. \quad (12)$$

Continuous forms of DAAs forms are thus obtained according to the order of both expansions used for the matching :

a) A 1-low, 1-high DAA (the continuous form of the DAA1) is obtained by enforcing $C_2 = H = T_1 = 0$ in the derivation of the previous subsection :

$$[I - ikT_0](\varphi) = T_0(g). \quad (13)$$

b) A 1-low, 2-high DAA (the continuous DAA2c) is obtained by enforcing $T_1 = 0$. It requires the operator Θ,

$$\Theta = T_0^{-1} + H,$$

to be invertible. It reads :

$$[I - ikT_0\Theta^{-1}T_0^{-1} - k^2T_0\Theta^{-1}](\varphi) = [T_0 - ikT_0\Theta^{-1}](g). \quad (14)$$

c) A 2-low, 1-high DAA is obtained by enforcing $H = 0$ in the previous subsection. It requires $(-iT_0^2 + T_1)$ to be invertible.

With this assumption this approximation reads :

$$[I - ikT_0 - k^2\,(T_0^2 + iT_1)](\varphi) = [T_0 - ik\,(T_0^2 + iT_1)](g). \quad (15)$$

d) Finally, a 2-low, 2-high DAA is obtained which requires the operators $(-iT_0^2 + T_1)$ and $(T_0^{-1} + H)$ to be invertible. It reads :

$$[I - ik(T_0 - iT_1H)\Theta^{-1}T_0^{-1} - k^2(T_0 + iT_1T_0^{-1})\Theta^{-1}](\varphi) = [T_0 - ik(T_0 + iT_1T_0^{-1})\Theta^{-1}](g). \quad (16)$$

4. CONCLUSION

Continuous forms of DAA's have been formally obtained and some geometrical assumptions which underlie their validity have been pointed out. It is clear that in this formal derivation only sufficient conditions (and not necessary) have been given. A complete and rigorous derivation would involve some difficult numerical analysis and geometry problems. Discretized forms may also be derived [3]. It is interesting to note that they slightly differ from the ones found in the literature [1].

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