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FRACTALNESS OF THE ISING CONFIGURATION

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Abstract. - The configurations of the Ising models are studied. At the critical point, the magnetization of each configuration has fractal nature with respect to the majority-rule scale transformation. This fractalness yields the hyperscaling relation and predicts the values of finite-size-scaling exponents in four- or more-dimensional lattices.

The scale invariance of the relevant system is essential to the second order transition [1]. It has been discussed for thermodynamic quantities such as free energy [2, 3]. In this paper, we will study the scale invariant geometric object behind the thermodynamic quantities. It is each configuration of the system appearing typically at the critical points [4, 5].

The Ising model on a \( d \)-dimensional hypercubic lattice whose size is \( L^d \) is considered here. The susceptibility per spin of this system, \( \chi_L \), behaves like

\[
\chi_L = \frac{1}{L^d} \left \langle (\Delta M)^2 \right \rangle \sim L^{2\gamma/v'},
\]

where \( \gamma \) denotes the critical exponent of the susceptibility and \( v' \) denotes the finite-size-scaling exponent [6-8] which is equal to the exponent of the correlation length, \( \nu \), in the dimensions up to the upper critical dimensionality.

The equation (1) shows the self-similar nature of the susceptibility at the critical point. The susceptibility is proportional to the average of the squared magnetization, \( \langle M^2 \rangle \) if the spontaneous magnetization is zero. At the critical point we have

\[
\langle M^2 \rangle \sim L^{d+\gamma/v'}.
\]

From this relation (2), it seems to be natural that the magnetization has a fractal nature, although the average is equal to 0. It was clarified that the magnetization of the typical configuration \( \Lambda \) at the critical point has the fractal nature with the following scale transformation, denoted by \( T_b \), which is the majority rule scale transformation with scale \( b \). Here the “typical” means that the emergence probability of the configuration is not too small in the configuration space. The \( T_b \) maps a configuration whose linear size is \( L \) to a configuration of linear size \( L/b \) as follows: the lattice is divided into \( (L/b)^d \) hypercubes of size \( b^d \) and if the magnetization of each box is positive (negative), the box is replaced by up (down) spin. If zero, a new spin state is selected with probability 1/2.

With this transformation \( T_b \), the magnetization of the typical configuration \( \Lambda \) behaves like

\[
M(T_b^n \Lambda) \sim \left ( \frac{L}{b^n} \right )^D, \quad (1 \ll b^n \ll L)
\]

as \( n \) is increased. \( M(\cdot) \) denotes the magnetization of the configuration.

In the previous paper [5], we have studied this nature with \( M(T_b \Lambda) \) for various scale of \( b \). The fractal nature at the critical point was clearly observed with this. The value of \( D \) is, however, slightly but definitely different from the expected value because of the transient-region effect of the fractal objects [9].

The value of \( D \) is expected to be

\[
D = \frac{1}{2} \left ( d + \frac{\gamma}{v'} \right ) = d - \frac{\beta}{v'},
\]

with comparing equations (2, 3) and with the similar argument about the spontaneous magnetization.

If we eliminate the variable \( D \) from (4), the scaling relation

\[
dv' = 2\beta + \gamma
\]

is obtained. In the case of \( \nu = \nu' \), this relation is the hyperscaling relation.

In the following, the fractal nature of the configuration at the critical point is studied with equation (3). The Monte Carlo method is used to study the Boltzmann distribution over the configuration space. Each configuration \( \Lambda \) appearing in the Monte Carlo sampling is analyzed with the variable,

\[
D_n(b) = \frac{\log \left ( M(T_b^{n-1} \Lambda)/M(T_b^n \Lambda) \right )}{\log b}.
\]

This \( D_n(b) \) is expected to approach \( d/2 \), \( D \) and \( d \) at \( T > T_c \), \( T_c \) and \( T < T_c \), respectively.

We have studied this \( D_n(b) \) for square and cubic Ising models. The scale factor \( b \) was 2.

Two-dimensional case.

The results are shown in figure 1. The values of \( D_n(2) \) approach the expected value 1.875 at the critical point. The small deviation due to the transient nature disappears which was observed in the previous paper [5].

Three-dimensional case.

The results are shown in figure 2. The values of \( D_n(2) \) also approach the value 2.48 expected from the
estimated critical exponents by high temperature expansion [10] and field theoretic calculation [11]. The values of $D_n$ are more sensitive to the temperature than those of the two-dimensional case.

Up to now, the values of $D_n (b)$ are statistically averaged ones. As an example, the distribution of the value $D_1 (3)$ of $64^2$ lattice is shown in figure 3. The distribution is very sharp Gaussian. The standard deviation $\sigma$ of this distribution is 0.02.

It was shown by Binder [12] that the distribution function of the block magnetization at the critical point, $P_L (M)$, has the scaling form,

$$P_L (M) = L^{-D} f (L^{-D} M), \quad (D = d - \frac{\beta}{\nu}) \quad (7)$$

where $L$ is the linear size of the block. Although this scaling suggests our fractalness theory, this scaling does not have the scale transformation. For example, if the scale transformation maps the configuration randomly, the deviation of $D$, $\sigma_{ac}$, is

$$\sigma_{ac}^2 = \langle D^2 \rangle - \langle D \rangle^2, \quad (8)$$

where

$$\langle D \rangle = \int_0^\infty dx \int_0^\infty dy \frac{\log (x/y)}{\log b} P_{KL} (y) P_L (x) = D \quad (9)$$

and

$$\langle D^2 \rangle = \int_0^\infty dx \int_0^\infty dy \left( \frac{\log (x/y)}{\log b} \right)^2 P_{KL} (y) P_L (x). \quad (10)$$

The estimated value of $\sigma_{ac}$ from the distribution function plotted in reference [12] is about 1.0 which is too large compared with the observed value 0.02.

The smallness of deviation will be useful to estimate the value of $\beta/\nu$ and $\gamma/\nu$ from the Monte Carlo simulation. Recently Miyashita [13] has applied our method successfully to his Monte Carlo study of the triangular lattice model with next nearest interaction.

If this fractal picture is valid in the dimensions higher than four, the finite-size-scaling exponent $\nu' \nu$ is expected to be $2/d$ from equation (5) and mean-field values of $\beta$ and $\gamma$. The relation $\nu' = 2/d$ for $d \geq 4$ is confirmed for $d = 5$ and conjectured by other authors [14].

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