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# IMPLICATIONS OF A CONSTITUTIVE MODEL FOR STRAIN LOCALIZATION

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**Résumé** – On donne une solution exacte et explicite qui décrit la formation d'une bande de glissement dans une théorie des matériaux rigide-plastiques qui s'adoucissent au de là d'une certain déformation. La solution donne une bande de glissement dont la largeur est déterminée par les paramètres du matériau, et non par les longueurs d'ondes spatiales associées aux pertubations de la configuration homogène initiale. Les lois de comportement mécanique de la théorie ne contiennent pas la vitesse de déformation explicitement, mais les paramètres de la théorie dépendent de cette vitesse dans les applications. La théorie doit fournir un modèle utile, bien qu'approximatif, pour le comportement des métaux ductiles dans les processus quasiadiabatiques.

**Abstract** – An explicit and exact solution describing the formation of a shear band is given in a theory of rigid-plastic materials that soften after a certain amount of deformation. The solution yields a shear band whose width is determined by material parameters, not by spatial wavelengths associated with perturbations of the homogeneous initial configuration. Although the constitutive relations of the theory do not contain the rate of deformation explicitly, the parameters in the theory will be rate-dependent in application. The theory is expected to supply a useful, albeit approximate, model for the mechanical behavior of ductile metals in quasiadiabatic processes.

## 1 – INTRODUCTION

As the yield stress of a ductile material is expected to decrease with increasing temperature, even if a material hardens with accumulated plastic strain in isothermal deformations, it can show apparent strain softening in processes rapid enough to be nearly adiabatic. Strain softening, whatever its cause, can destabilize homogeneous deformations. Hence, one expects that when the strain accumulated in an initially homogeneous motion attains a level at which

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subsequent flow must decrease the yield stress, further strain will be localized in narrow bands. This expectation is the basis of the accepted explanation of the shear bands observed when metals are deformed at ballistic rates.[1] A goal of our research on strain localization has been the development of physically acceptable constitutive relations for which the problem of producing exact solutions showing shear bands is tractable. In this note we summarize some of our results.[2]

Like other deformations, the formation of a shear band is influenced by the viscous stresses and internal relaxation phenomena that accompany plastic flow, but such intrinsic rate effects do not appear vital for occurrence of the phenomenon. Essential to a theory of shear bands is an expression of the ability of plastic flow to lower the stress required for further flow. In the theory we have been developing, this is expressed by using constitutive relations that describe a rigid-plastic material with a yield function $\phi$ that, in motions of shear, depends on the total variation $\bar{\gamma}$ of the shear strain $\gamma$ and decreases with increasing $\bar{\gamma}$ when $\bar{\gamma}$ exceeds a critical value $\bar{\gamma}_m$. Of importance for the construction of a theory that can describe such severe manifestations of strain localization as shear bands is the need to render mathematical the physical idea that when strain varies rapidly from place to place this variation must itself influence the stress. In regions in which $\bar{\gamma}$ has large spatial derivatives of order greater than one, there should be a force playing a role similar to that of the capillary forces that act to oppose increases of interfacial area in theories of phase transformation and crack propagation. This force is expected to mollify the growth of inhomogeneities without totally suppressing them. We have found that the simplest way to account for such a force is to add a term linear in the second spatial derivative of $\bar{\gamma}$ to the classical expression for the stress in a flowing rigid-plastic material.

With the above observations as background, we now state, in the framework of unidimensional motions of shear, the constitutive assumptions and field relations we have studied. The usefulness of these assumptions should be judged by the simplicity and success with which they account for the important features of the phenomena they are intended to explain.

## 2 – CONSTITUTIVE ASSUMPTIONS

The present discussion is confined to motions for which the flow is in the $y$-direction of a Cartesian system, and the displacement $u$ in that direction is a function of $x$ and $t$, with $t$ the time that has elapsed since the body was in a stress-free undistorted reference state with $u = 0$. The *shear strain* $\gamma$ and the *rate of shear* $\gamma_t$ are

$$\gamma(x,t) = u_x(x,t) = \frac{\partial}{\partial x} u(x,t), \tag{2.1}$$

$$\gamma_t(x,t) = \frac{\partial}{\partial t} \gamma(x,t). \tag{2.2}$$

For each $t \geq 0$ and each $x$, the *accumulated shear strain* $\bar{\gamma}(x,t)$ is the total variation of the shear strain at $x$, considered a function on $[0,t]$:

$$\bar{\gamma}(x,t) = \int_0^t \left| \frac{\partial}{\partial \sigma} \gamma(x,\sigma) \right| d\sigma. \tag{2.3}$$

The *shear stress* $\tau$ is the $(y,x)$-component of the Cauchy stress tensor.

Each material of the class we consider is characterized by a positive constant $c$ and a *yield function* $\phi$ that is positive on an interval of the form $[0, \bar{\gamma}_F)$;[(2)] these enter the relations:

$$\text{if } \gamma_t = 0, \text{ then } |\tau| \leq \phi(\bar{\gamma}); \tag{2.4a}$$

$$\text{if } \gamma_t \neq 0, \text{ then } |\tau| \geq \phi(\bar{\gamma}) \text{ and } \tau = [\phi(\bar{\gamma}) - c\bar{\gamma}_{xx}] \frac{\gamma_t}{|\gamma_t|}. \tag{2.4b}$$

The following assertions are easily derived consequences of the basic constitutive assumptions (2.4):

$$\text{if } \gamma_t \neq 0 \text{ and } \tau\gamma_t \geq 0, \text{ then } \bar{\gamma}_{xx} \leq 0; \tag{2.5}$$

$$\text{if } |\tau| < \phi(\bar{\gamma}), \text{ then } \gamma_t = 0; \tag{2.6}$$

$$\text{if } |\tau| > \phi(\bar{\gamma}), \text{ then } \gamma_t \neq 0 \text{ and } \tau = [\phi(\bar{\gamma}) - c\bar{\gamma}_{xx}] \frac{\gamma_t}{|\gamma_t|}. \tag{2.7}$$

The idea that, after a certain amount of accumulated strain, plastic flow can lower the stress required for further flow is made mathematical by assuming that $\phi$ attains a maximum at a point $\bar{\gamma}_m$ in $(0, \bar{\gamma}_F)$. The simplest smooth function $\phi$ with $\phi(0) > 0$, $\phi'(0) > 0$, $\phi'(\bar{\gamma}_m) = 0$, and $\phi''(\bar{\gamma}_m) < 0$ is the quadratic,

$$\phi(\bar{\gamma}) = \phi_m - \alpha\,(\bar{\gamma} - \bar{\gamma}_m)^2\,, \tag{2.8}$$

in which $\alpha$, $\bar{\gamma}_m$, and $\phi_m$ are material constants[(3)] obeying

$$\alpha > 0, \qquad \bar{\gamma}_m > 0, \qquad \phi_m > \alpha\bar{\gamma}_m^2. \tag{2.9}$$

Of course, $\phi_m$ is the maximum yield stress, $\phi(\bar{\gamma}_m)$, and is positive. With

$$\bar{\gamma}_F = \bar{\gamma}_m + \sqrt{\phi_m/\alpha}, \tag{2.10}$$

we have

$$\lim_{\bar{\gamma} \to \bar{\gamma}_F} \phi(\bar{\gamma}) = 0, \tag{2.11}$$

and $[0, \bar{\gamma}_F)$ is the interval of values on which $\phi$ is positive. As $\bar{\gamma}$ approaches $\bar{\gamma}_F$, the largest value of $|\tau|$ that the material can sustain without flowing approaches zero. A body with $\bar{\gamma} = \bar{\gamma}_F$ at a plane $x$ = constant would appear to have zero strength for shear in that plane and would be said to have "failed" or "lost coherence".

If we assume that the shear stress $\tau$ is a function of $x$ and $t$, that the pressure in the sample does not vary with $z$, that the applied body forces are conservative, and that inertia can be neglected, balance of forces yields

$$\tau(x,t) = \tau^\circ(t) - b(t)x. \tag{2.12}$$

(2) The number $\bar{\gamma}_F = \sup\{\bar{\gamma} \mid \phi(\bar{\gamma}) > 0\}$ is called the *accumulated strain at failure*. There is no *a priori* reason to suppose that $\bar{\gamma}_F$ is not infinite, *i.e.*, $\phi$ may be positive on $[0, \infty)$, but when $\phi$ has the special form seen in equation (2.8) with $\alpha$, $\gamma_m$, and $\phi_m$ as in (2.9), $\bar{\gamma}_F$ is finite.

(3) In practice, $\alpha$, $\bar{\gamma}_m$, and $\phi_m$ will depend on the rate of deformation and the initial temperature.

We may suppose that the material lies between and adheres to plates at the planes $x = d/2$ and $x = -d/2$.

In the following section we give an explicit solution we have obtained[2] describing the nonsteady shearing motion generated by moving the bounding plates in opposite directions in the absence of body forces and pressure gradients; for such a motion $b(t)$ is zero, and hence the shear stress is a function of time alone:

$$\tau(x,t) = \tau^\circ(t). \tag{2.13}$$

The solution is exact under the assumption that the separation of the bounding plates is so large that we may concentrate attention on regions near the plane $x = 0$ and take $d$ to be infinite.

We have also treated channel flow, that is, a flow for which $d$ is finite, the plates are stationary, and it is the application of a driving force that causes the material to move in the $y$-direction. Flow fields that are symmetric about the plane $x = 0$ and for which $\tau^\circ = 0$, so that (2.12) becomes $\tau = -b(t)x$, with $b(t)$ the driving force per unit volume, are given in Hodgdon's doctoral thesis[3] and in a paper now in press[4]. The space alloted here does not permit a discussion of channel flow.

When $\phi$ has the form (2.8), the constitutive assumptions (2.4) and the field equation (2.12) are together equivalent to the assertion that for all $x$ and $t$:

$$\text{if } \gamma_t(x,t) = 0, \text{ then } |\tau^\circ(t) - b(t)x| \le \phi_m - \alpha\left[\bar\gamma(x,t) - \bar\gamma_m\right]^2; \tag{2.14a}$$

$$\begin{aligned}\text{if } \gamma_t(x,t) \ne 0, \text{ then } & |\tau^\circ(t) - b(t)x| \ge \phi_m - \alpha\left[\bar\gamma(x,t) - \bar\gamma_m\right]^2 \\ & \text{and } \tau^\circ(t) - b(t)x = \left[\phi_m - \alpha\left[\bar\gamma(x,t) - \bar\gamma_m\right]^2 - c\bar\gamma_{xx}(x,t)\right]\frac{\gamma_t}{|\gamma_t|}.\end{aligned} \tag{2.14b}$$

We seek motions, compatible with the initial condition $u(x,0) = \bar\gamma(x,0) = 0$, that obey (2.14) for all $t$ in $[0,t_F)$ and all $x$; here $t_F$ is the *time of failure* in the sense that for some value $x_F$ of $x$,

$$\lim_{t\to t_F} \bar\gamma(x_F,t) = \bar\gamma_F, \tag{2.15}$$

and $0 \le \bar\gamma(x,t) < \bar\gamma_F$ for all $t$ in $[0,t_F)$ and all $x$.

## 3 – THE SHEAR-BAND SOLUTION

For the case in which $b = 0$ so that (2.13) holds, we have found, for $d$ infinite, an exact solution of the relations (2.14) obeying the following conditions: (a) $\gamma_t$ agrees in sign with $\tau^\circ$, *i.e.*, for each $x$ and $t$, $\tau^\circ(t)\gamma_t(x,t) \ge 0$; (b) for each $t$, $\bar\gamma(x,t)$ is a bounded even function of $x$ which attains its maximum at $x = 0$; (c) $|\tau^\circ(\cdot)|$ is a continuously differentiable function; (d) $|\tau^\circ(t)|$ increases from 0 to $|\tau^\circ(t_m)| = \phi_m$ as $t$ varies over an interval $[0,t_m]$ with $t_m > 0$; $|\tau^\circ(t)|$ decreases with increasing $t$ for $t > t_m$;[(4)] (e) $\gamma(x,t)$ is continuous in $x$ and $t$; and (f) $\gamma_t(x,t)$, $\bar\gamma_x(x,t)$ and $\bar\gamma_{xx}(x,t)$ are bounded and piecewise continuous in $x$ and $t$. In this solution, which describes the shearing of a large volume of material between widely separated moving plates,

(4) We assume that $d|\tau^\circ(t)|/dt$ is (strictly) positive for $t$ in $[0,t_m)$ and (strictly) negative for $t$ in $(t_m,t_F)$.

the motion is homogeneous up to time $t_m$ after which strain concentration occurs near the center plane, $x = 0$.

Let $t_0$ be the time at which $|\tau^\circ(t)|$ first attains the value $\phi(0)$. Our solution is such that $u(x,t) = \gamma(x,t) = \bar{\gamma}(x,t) = 0$ for all $x$ and each $t$ in $[0, t_0]$, *i.e.*, there is no motion until $|\tau^\circ|$ exceeds $\phi(0)$. For $t$ in $(t_0, t_m]$ the motion is one of homogeneous shearing with, for all $x$, $\bar{\gamma}(x,t) = \bar{\gamma}^\circ(t)$, where $\bar{\gamma}^\circ(t)$ is the smaller of the two positive roots of the equation $\phi(\bar{\gamma}^\circ(t)) = |\tau^\circ(t)|$:

$$\bar{\gamma}(x,t) = \bar{\gamma}_m - \left[\frac{\phi_m - |\tau^\circ(t)|}{\alpha}\right]^{1/2}, \qquad \text{for } t_0 \leq t \leq t_m \text{ and all } x. \tag{3.1}$$

When $|\tau^\circ|$ decreases from its maximum value $\phi_m$, the strain field does not remain homogeneous, but starts to show a shear band. That is, for each $t > t_m$ there is a band, $-x^*(t) < x < x^*(t)$, such that $\gamma_t(x,t) \neq 0$ for $x$ in this band, *i.e.*, for $|x| < x^*(t)$, and $\gamma_t(x,t) = 0$ for $x$ outside, *i.e.*, for $x > |x^*(t)|$. The half-width $x^*(t)$ of this band of active shearing decreases with increasing $t$ and is given by the formula(5)

$$x^*(t) = 0.88952 c^{1/2} \alpha^{-1/4} \left[\phi_m - |\tau^\circ(t)|\right]^{-1/4}, \qquad (t > t_m). \tag{3.2}$$

For each $t > t_m$, $\bar{\gamma}_{xx}(x,t) < 0$ and $\bar{\gamma}_t(x,t) > 0$ wherever $|x| < x^*(t)$, and

$$\lim_{|x| \to x^{*-}} \bar{\gamma}_{xx}(x,t) = \lim_{|x| \to x^{*-}} \dot{\bar{\gamma}}_t(x,t) = 0. \tag{3.3}$$

In the shear band, we have the following expression(6) for $\bar{\gamma}$:

$$\bar{\gamma}(x,t) = \bar{\gamma}_m + \delta(t)^{1/2} \left[\eta - \lambda^2 \frac{1 - \mathrm{cn}(v(t)x|m)}{1 + \mathrm{cn}(v(t)x|m)}\right], \qquad \text{for } t > t_m \text{ and } |x| \leq |x^*(t)|; \tag{3.4}$$

here

$$\delta(t) = \frac{\phi_m - |\tau^\circ(t)|}{\alpha}, \qquad v(t) = \lambda \delta(t)^{1/4} (2\alpha/3c)^{1/2}. \tag{3.5}$$

The dimensionless quantity $\delta(t)$ is a measure of the decline of the magnitude of the shear stress from its maximum value; cn is the indicated Jacobian elliptic function(7) with $m = 0.979681$. The numbers $\lambda$ and $\eta$ arise in the theory of elliptic integrals and are related to $m$ by the equations(8)

$$\lambda^4 = 3(\eta^2 - 1), \qquad m = \frac{1}{2} + \frac{3\eta}{4\lambda^2}; \tag{3.6}$$

and hence $\eta = 2.324236$, and $\lambda = 1.90632$. For $\bar{\gamma}$ at points outside the shear band, we have(9)

$$\bar{\gamma}(x,t) = \bar{\gamma}_m + \frac{0.79142c}{\alpha |x|^2}, \qquad \text{for } t > t_m \text{ and } x \geq |x^*(t)|. \tag{3.7}$$

(5) Ref. 2, eq. (3.49).

(6) Ref. 2, eq. (3.51).

(7) See, *e.g.*, Milne-Thomson[5]. Here $m = k^2$ with $k$ the modulus of the elliptic function.

(8) The present $\eta$ is written $\eta_0$ in Ref. 2.

(9) Ref. 2, eq. (3.59).

The formula (3.2) for $x^*(t)$ implies that when $|\tau^\circ(t)|$ is close to $\phi_m$, *i.e.*, when $\delta(t)$ is small, the band of active shearing (where $\gamma_t \neq 0$) is wide; in the limit $t \to t_m^+$ this band extends from $x = -\infty$ to $x = \infty$. As $|\tau^\circ(t)|$ drops below $\phi_m$ the band narrows rapidly, but, as $|\tau^\circ(t)|$ continues to drop, the band width, $2x^*$, eventually diminishes slowly, *i.e.*, remains nearly constant, for $x^*$ varies with $\tau^\circ$ as $(\phi_m - |\tau^\circ|)^{-1/4}$. While this is happening, the accumulated shear at the center of the band, $\bar{\gamma}(0,t)$, is increasing with time as $[\phi_m - |\tau^\circ(t)|]^{1/2}$, for (3.4) and (3.5) yield,(10)

$$\bar{\gamma}(0,t) - \bar{\gamma}_m = \eta\delta(t)^{1/2} = 2.3242\alpha^{-1/2}\left(\phi_m - |\tau^\circ(t)|\right)^{1/2}. \tag{3.8}$$

Because, for each $t$, $\bar{\gamma}(0,t)$ is the maximum value of $\bar{\gamma}$ in the body, $\bar{\gamma}_F$ is attained at $x = 0$, *i.e.*, $x_F = 0$. It follows from (2.10) and (3.8) that $|\tau^\circ(t)|$ at the moment of failure, $t_F$, is(11)

$$|\tau^\circ(t_F)| = (1 - \eta^{-2})\phi_m = 0.8149\phi_m. \tag{3.9}$$

At that moment, the half-width of the band is, by (3.2),

$$x^*(t_F) = 1.356c^{1/2}(\alpha\phi_m)^{-1/4}. \tag{3.10}$$

The assumed smoothness of $|\tau^\circ|$ as a function of $t$ does not preclude cases in which $\tau^\circ$ suffers sudden jumps from $\tau^\circ(t)$ to $-\tau^\circ(t)$. However, for metals undergoing deformation at ballistic rates, the important case is that in which $\tau^\circ$ does not change sign. If $\tau^\circ$ is never negative, then (3.1) and (2.3) yield $\gamma(x,t) = \bar{\gamma}(x,t)$, and if we choose a frame so that the displacement $u$ is always zero at $x = 0$, we have

$$u(x,t) = \int_0^t \gamma(\xi,t)d\xi. \tag{3.11}$$

It follows from (3.4)–(3.7) and (3.11), that, for $t > t_m$,(12)

$$u(x,t) = \bar{\gamma}_m x + \delta(t)^{1/2}\left[\eta x - \lambda^2 x + \frac{2\lambda^2}{v(t)}\left[E\big(v(t)x\big|m\big) - \frac{\operatorname{sn}\big(v(t)x\big|m\big)\operatorname{dn}\big(v(t)x\big|m\big)}{1+\operatorname{cn}\big(v(t)x\big|m\big)}\right]\right],$$
$$\text{for } 0 \le x \le x^*(t); \tag{3.12a}$$

$$u(x,t) = \delta^{1/4}(t)(c/2\alpha)^{1/2}\left[1.25797\bar{\gamma}_m\delta(t)^{-1/2} + 2.30590\right], \qquad \text{for } x = x^*(t); \tag{3.12b}$$

$$u(x,t) = u\big(x^*(t),t\big) + \bar{\gamma}_m\left[x - x^*(t)\right] + 0.79124\frac{c}{\alpha}\left[\frac{1}{x^*(t)} - \frac{1}{x}\right], \qquad \text{for } x \ge x^*(t); \tag{3.12c}$$

$$u(-x,t) = -u(x,t), \qquad \text{for all } x \ge 0. \tag{3.12d}$$

(10) Ref. 2, eq. (3.53).

(11) Ref. 2, eq. (3.54).

(12) Ref. 2, eq. (3.67). In order to write that expression in its present form, we use the value 1.25797 for the number $\xi^*$ defined in Ref. 2 and find that this yields $\operatorname{cn}(\lambda 3^{-1/2}\xi^*|m) = 0.465849$ and $E(\lambda 3^{-1/2}\xi^*|m) = 0.89003$. $E(\,\cdot\,|m)$ is given by an elliptic integral of the second kind; *i.e.*, $E(w|m) = \int_0^\psi (1 - m\sin^2\theta)^{1/2}d\theta$, with $\psi = \sin^{-1}\left(\operatorname{sn}(w|m)\right)$.

Such displacement fields are shown in the accompanying Figure, where $x$ is the ordinate, $u(x,t)$ is the abscissa, and the material parameters have been given values, $\alpha = 1$, $\phi_m = 10$ $\bar{\gamma}_m = 0.05$, $c = 0.1$. The graphs seen there may be thought of as pictures of a scribe line that lies along the $x$-axis when $t = 0$. The solid line shows the scribe line at the time $t_m$ at which $\tau^\circ(t_m) = \phi_m$; at that time the line is straight, as it is at earlier times; at later times it is curved. The dotted curve depicts the deformed scribe line at the time $t_F$ at which the material fails.

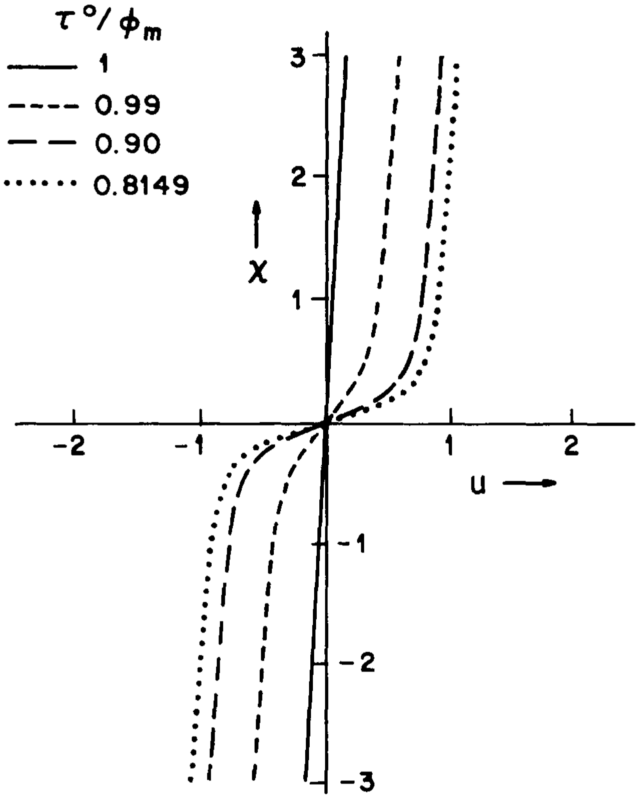


**Figure.** Graphs, based on equations (3.12a)–(3.12d), showing the displacement $u$ in the $y$-direction as a function of $x$ at various times during shearing between widely separated parallel plates.

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