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SUPERRADIANCE THEORY AND X-RAYS LASERS EXTENDING THE SEMI-CLASSICAL MODEL TO REALISTIC SITUATIONS

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As shown in A. Crubelier's introductory paper, spontaneous emission of a collective ensemble of atoms, i.e. superradiance (S.R.), or superfluorescence, is very different from "ordinary" spontaneous emission.

This phenomenon, initially pointed out by Dicke [1], has been extensively studied both experimentally [2][3][4] and theoretically.

This paper makes implicit reference to [5] and to the various papers which describe superradiance using the Bloch-Maxwell equations formalism [6][7]. These equations, which couple the atomic dipoles to a classical electromagnetic field, are well adapted to describe propagation and amplification of a macroscopic classical field in its full complexity, i.e. including propagation, transverse effect (diffraction and off-axis emission) and/or other effects as dephasing processes (Doppler effect)...

This semi-classical formalism is unable to explain how a collection of N atoms, initially prepared in the upper level of a two-level transition (superradiant initial state), are able:

i) to start radiating

ii) to build up an electromagnetic field with a given phase (this implies a symmetry breakdown).

Actually, this initial symmetry breakdown is very well understood in the context of the superradiance theory [8][9]. This paper will try to explain this point, in an original way (to our knowledge), for people who are not familiar with superradiance theory. This will be done in three steps:

i) Bloch-Maxwell equations are presented in the full three-dimension space case (§1). By the way, three-dimensional transverse effects are described (§2).
ii) The initial symmetry breakdown is studied for a superradiant emission triggered by a blackbody field. This is done in the three-dimensional space case, conceptually more simple, and less ambiguous than the one-dimensional one. It is shown that blackbody noise can be described by a random boundary field, whose statistical properties are easily determined (§3).

iii) Spontaneous emission and blackbody noise are related together by a scaling factor which is determined in a simple case. Extending the result to more complex situations is straightforward. The spontaneous emission versus blackbody scaling factor depends on the field operator (a and a†) order. Two independent sets of boundary conditions, corresponding respectively to normal and antinormal ordered operators, are thus able to describe spontaneous emission noise. As the superradiant emitted field is macroscopic, it does not depend on the field operator order, and these two sets of boundary conditions are strictly equivalent (§4).

The last section of this paper (§5) describes briefly how to deal with dephasing processes as Doppler effects. Connection between superradiance and amplified spontaneous emission is made.

1. The Bloch–Maxwell equations for superradiance

Fig. 1

In the whole paper, we will consider an ensemble of N two level identical atoms. The upper and lower states of each atom are called 1 and 2 (see fig. 1) respectively. They are separated by an energy interval ħω₀ (λ₀=2πc/ω₀ is the wavelength of the superradiant transition). In the case of superradiance, a total population inversion is initially created between 1 and 2, for example by pumping the atoms from a ground state g to level 1.
The active medium is supposed to have a pencil shape, whose axis is \(z\), and the forward emitted field is only considered (see fig. 2). With these hypotheses, the wave vector \(\hat{k}\) of the S.R.-emitted field is close to \(\hat{k}_0\), with \(\hat{k}_0 = \omega_0/c.\hat{u}_z\). The projections \(\xi^\pm\) and \(\rho^\pm\):

\[
\tilde{\mathbf{E}} = \left[ \tilde{\xi}^+ \exp(-i(\omega_0 t - \hat{k}_0 \cdot \hat{r})) + \tilde{\xi}^- \exp(i(\omega_0 t - \hat{k}_0 \cdot \hat{r})) \right] \cdot \hat{u}_x
\]

\[
\tilde{\mathbf{P}} = \left[ \tilde{\rho}^+ \exp(-i(\omega_0 t - \hat{k}_0 \cdot \hat{r})) + \tilde{\rho}^- \exp(i(\omega_0 t - \hat{k}_0 \cdot \hat{r})) \right] \cdot \hat{u}_x
\]

of the electric field \(\tilde{\mathbf{E}}\), and of the atomic polarization \(\tilde{\mathbf{P}}\) on the \(\hat{k}_0\) plane wave are the then slowly varying in space and in time. These quantities are often called "slow varying envelopes".

In eq. (1), we consider that the electric field \(\tilde{\mathbf{E}}\) is purely transverse \((\tilde{\mathbf{E}}/\hat{u}_x)\): we neglect dipole-dipole interactions \([10]\). This is valid when the distance between neighbour atoms is larger than the wavelength \(\lambda\).

The atomic polarisation \(\rho\) is related to the off-diagonal components \((\rho_{12})_i\) and \((\rho_{21})_i\) of the \(i^{th}\) atom density matrix \(\rho_i\) by:

\[
\tilde{\mathbf{P}} = d. \sum_{\text{atoms}} \left( (\rho_{12} + \rho_{21})_i \delta(\hat{r} - \hat{r}_i) \right) \cdot \hat{u}_x
\]

In this equation, \(d\) is the atomic dipole of the 1→2 transition, and \(\hat{r}_i\) is the location of the \(i^{th}\) atoms.

The field \(\tilde{\mathbf{E}}^\pm\), the polarization \(\tilde{\mathbf{P}}^\pm\) and the inversion density \(\mathcal{N}\):

\[
\mathcal{N} = \frac{1}{2} \sum_{\text{atoms}} \left( (\rho_{11} - \rho_{22})_i \delta(\hat{r} - \hat{r}_i) \right)
\]

(where \(\rho_{11}\) and \(\rho_{22}\) are the individual diagonal elements of the density matrix) are coupled together by the so-called Bloch-Maxwell equations \([11]\). We use here the notations of \([5]\).
Eq. (5) is Maxwell's equation for the envelopes $\xi^+$ and $\eta^+$. It is equivalent to:

$$\frac{\partial^2 \xi^+}{\partial t^2} + \frac{1}{c^2} \frac{\partial \xi^+}{\partial z} - \frac{1}{2k_0} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \xi^+ = \frac{i\omega}{2\varepsilon_0 c} \mathcal{P}^-$$

$$\frac{\partial}{\partial t} \mathcal{N} = \frac{1}{\hbar} \left[ \mathcal{P}^+ \xi^+ - \mathcal{P}^- \xi^- \right]$$

$$\frac{\partial \mathcal{P}^+}{\partial t} = \frac{2i\hbar}{\hbar^2} \xi^- \mathcal{N}$$

Eq. (5) is Maxwell's equation for the envelopes $\xi^+$ and $\eta^+$. It is

equivalent to:

$$\text{curl} (\text{curl} E) = \frac{1}{c^2} \frac{\partial}{\partial t^2} \left[ \frac{\partial}{\partial t} + \frac{1}{\varepsilon_0} \mathcal{P}^- \right]$$

The first $\partial^2 / \partial z^2$ term of eq. (5) describes the longitudinal amplification of the field. The second one $(1/c^2) (\partial / \partial t) \xi^+$ corresponds to the retardation effect. This term can be squeezed by replacing $t$ by $t-z/c$, and we will forget it in the following. The third term:

$$\left[ - \frac{1}{2k_0} \right] \left[ \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right] \xi^+$$

is dealing with the finite transverse size of the active medium, i.e. diffraction and off-axis emission. Analysis of these effects will be done in the next section (section 2).

Eqs. (6) and (7) are Bloch's equations and describe the atomic evolution in the external field $\xi^+$. These equations are local and correspond to Schrödinger's one-atom equation:

$$i\hbar \frac{d}{dt} \rho = \left[ H, \rho \right]$$

Envelopes $\xi^+, \eta^+$ and $\mathcal{N}$ are used just for convenience.
Bloch–Maxwell equations introduced here are semi-classical equations coupling classical quantities, i.e. expectation value of operators, or matrix element of the individual atom density matrix. These equations can also be considered as quantum equations coupling operators. In this latter case, $\zeta^\pm$, $\mathcal{P}^\pm$ and $\mathcal{W}$ are operators which do not necessarily commute: The order of the various factors in operator products as $\mathcal{P}^+ \zeta^+$ is essential.

Bloch–Maxwell boundary conditions for superradiance are:

\begin{align}
\mathcal{P}^\pm (t = 0) &= 0 \\
\zeta^\pm (z = 0) &= 0
\end{align}

Bloch–Maxwell semi-classical equations are thus unable to explain the initial symmetry breakdown, which makes a non-vanishing emission build up, with a given phase. This initial symmetry breakdown will be studied in sections 3 and 4, either by using quantum Bloch–Maxwell equations (section 4), or by introducing in semi-classical equations the ad-hoc boundary value for $\mathcal{P}^\pm (t=0)$ and/or $\zeta^\pm (z=0)$ (sections 3 and 4).

Combining eqs (6) and (7):

\[ \mathcal{W}^2 + d. \left[ \mathcal{P}^+ \cdot \mathcal{P}^- + \mathcal{P}^- \cdot \mathcal{P}^+ \right] = cte \]

This conservation law yields to introduce, for the semi-classical description, the angular local variables $\Theta(x,y,z,t)$ and $\Psi(x,y,z,t)$:

\[ \mathcal{W} = \mathcal{W}_0 \cos \Theta \]

\[ \mathcal{P}^+ = d. \mathcal{P}_0 \sin \Theta \exp (i\Psi) \]

The state of the atomic system can thus be represented by a vector (Bloch's vector) \([\vec{z}]\) with spherical coordinates $\Theta$ and $\Psi$. $\Theta(t=0)$ is generally called the tipping angle \([\theta]\).

2. Finite transverse size of the atomic sample and diffraction

Except in very special cases, Bloch–Maxwell equations have no analytical solutions, and a numerical computation is necessary. Let's discuss briefly how to do this and particularly how to deal with the
transverse variation, in the X and Y directions of the field $\mathcal{E}^\pm$ and atomic dipole $\mathcal{D}^\pm$.

The first step, for solving Bloch–Maxwell equations, is to use a four-dimensional grid in space and time to discretize Bloch–Maxwell equation. In the following, we will call $\Delta x$, $\Delta y$, $\Delta z$, $\Delta t$ the space and time increments (see fig. 3).

![Fig. 3](image)

In Maxwell's equation (eq. 5), the diffraction term:

$$\left[ -\frac{1}{2\kappa_0} \right] \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{bmatrix} \mathcal{E}^+,$$

is difficult to discretize as it implies a second order derivative. Specific numerical techniques are necessary.

A first technique, which uses a non-linear grid, which varies in time, has been developed by Mattar [13].

A second one [14] makes a transverse Fourier transform of the $\mathcal{E}^\pm$ and $\mathcal{D}^\pm$ envelopes:

$$\mathcal{E}^\pm(x,y...) \rightarrow \mathcal{E}^\pm(k_x,k_y...) = \int dx \int dy \exp i(k_x x + k_y y) \mathcal{E}^\pm$$  \hspace{1cm} (16)

$$\mathcal{D}^\pm(x,y...) \rightarrow \mathcal{D}^\pm(k_x,k_y...) = \int dx \int dy \exp i(k_x x + k_y y) \mathcal{D}^\pm$$  \hspace{1cm} (17)

and yields to a numerically stable Maxwell's equation which can be discretized easily:

$$\left[ \frac{\partial}{\partial z} + i \left( \frac{k_x^2 + k_y^2}{2\kappa_0} \right) \right] \mathcal{E}^+ = -\frac{\omega_0}{2\varepsilon_0} \mathcal{D}^+$$  \hspace{1cm} (18)
Numerical simulation using this technique implies big computers, as two-dimensional fast Fourier transforms from $\hat{\mathcal{E}}^+$ to $\hat{\mathcal{E}}^+$ and from $\hat{\mathcal{P}}^+$ to $\hat{\mathcal{P}}^+$ are necessary for each step in time and in space (3-dimensional space).

Note that the transverse Fourier field $\hat{\mathcal{E}}^+(z=L)$ describes the far-field-emitted field, and is physically much more interesting than the local fields $\hat{\mathcal{E}}^+(z)$.

Once discretization is performed, the initial boundary values at $z=0$ and $t=0$ must be determined. This will be discussed in the following sections (§3 and 4). Bloch-Maxwell equations are intrinsically rather stable and numerical simulation generally does not cause trouble except in computing time.

Let us now discuss briefly about superradiance transverse effects (diffraction and off-axis emission), and how they are related to the transverse size of the atomic sample [13]. Qualitative analysis of these effects is made on figures 4 and 5. The initial atomic inversion density is supposed to have a transverse Gaussian profile:

$$\mathcal{N}(t=0,x,y,z) = \mathcal{N}_0 \cdot \exp \left[ -\frac{x^2 + y^2}{w_o^2} \right]$$

which corresponds to the profile of the pump laser used to prepare the atoms in the initial level 1 (see fig. 1).

For this qualitative analysis, we have considered that the superradiant emission is triggered by a small homogeneous atomic polarization with the following boundary conditions:

$$\int \xi(t=0) = 4 \cdot 10^{-4} \mathcal{N}_0$$

$$\frac{\partial \xi}{\partial t}(t=0) = 0$$
Fig. 4 shows the superradiant emitted intensity when the diffraction term in Maxwell’s equation is neglected. This corresponds to the case of an active medium with a large transverse size $w_0$, i.e. with a large Fresnel’s number ($F \gg 1$):

$$F = \frac{\left(\Pi \log 2 \right) w_0^2}{L \lambda}$$  \hspace{1cm} (22)

Fig. 4a shows the on-axis emitted field. Off-axis emitted field has the same shape but the scaling factor $T_R$ is different (see eq. 42). The characteristic “ringing” of fig. 4-a is washed out when the total superradiant emitted field is considered (fig. 4-b). The difference between fig. 4-a and 4-b is simply due to a space averaging effect.

Figure 5 shows the total superradiant emitted field for various
Fresnel's numbers $F = \infty, 11, 1.0, 0.4$ and $0.1$. Difference with the $F = \infty$ curve is due to diffraction. When the Fresnel number is larger than 0.4, the superradiant emission is slightly affected by diffraction. When Fresnel's number is smaller ($F \leq 0.1$), superradiance takes more time to build up.

The results on figs 4 and 5 correspond to a transverse homogeneous boundary condition (eq. 20), unable to describe off-axis emission. Off-axis emission, which becomes important for large $F$ Fresnel's numbers, is due to the boundary condition transverse symmetry breakdown. This is related to the blackbody, and/or spontaneous emission fluctuations, which will be studied in the next two sections (3 and 4).

3. **Blackbody fluctuation and superradiance triggering**

In order to understand the initial superradiant emission build-up, we will consider first the case of an atomic system coupled to blackbody field whose effect is much larger than spontaneous emission. This happens when the temperature $T$ is large enough so that:

$$\kappa T \gg \hbar \omega_0$$

Spontaneous emission can thus be neglected and superradiance is simply due to the incoming blackbody field. It is then possible to deal with superradiance triggering, in the semi-classical model, by considering the effect of a non-vanishing blackbody random field:

$$\xi^+ (z=0) \neq 0$$

Due to the four-dimensional numerical grid, the blackbody field $\xi^+ (z=0)$ must be averaged both in space: on the $[x, x+\Delta x]$ and $[y, y+\Delta y]$ intervals, and in time: on the $[t, t+\Delta t]$ interval. Let us call $\overline{\xi^+}$ the result of this field average. This three-dimensional average must be calculated for each step corresponding to the boundary condition (eq. 23), i.e. for $z=0$ and $x, y, t \neq 0$.

As the thermal field is random with a non-vanishing correlation length and correlation time, the average fields $\overline{\xi^+}$ obey to a Gaussian statistics and are statistically independent from each other:

$$\overline{\xi^+} = \overline{\xi_1^+} \cdot \exp i\psi$$

(24)
\( \Psi \) is an uniformly distributed random phase and \( | \xi \rangle \) is a random amplitude whose probability \( p(| \xi |^2) \) is:

\[
p (| \xi |^2) \propto \exp (-| \xi |^2 / \xi_o^2)
\]  

The random field mean square value \( \xi_0 \) can be easily deduced from the electromagnetic energy. \( \xi_0 \) is the field amplitude of \( n_\psi \) photons in a \( \Delta x. \Delta y. \Delta t \) box:

\[
n_\psi, n_\omega = \frac{1}{2} \epsilon_o \Delta x. \Delta y. \Delta t \xi_o^2
\]

where \( n_\psi \) is the Boltzmann's factor:

\[
n_\psi = \left[ \exp (n_\psi \omega / K T) - 1 \right]^{-1} \approx KT / \hbar \omega
\]

In the next section, we will see, on an example, that spontaneous emission and blackbody field act very similarly on the initially inverted atom, and we will determine the relative weight of the spontaneous emission, and blackbody noise.

4. Spontaneous emission fluctuation

In order to describe spontaneous emission, the electromagnetic field \( \xi^\pm \) and the atomic polarization \( j^\pm \) have to be considered as operators:

\[
\xi^\pm = \int d^3 k. N(k). i \omega a_k^\pm \exp \pm i \left[ \omega_o t + (k-\xi_0). \vec{r} \right]
\]

\[
j^\pm = d. \sum_i r_i^\pm g(i, \vec{r}_i). \exp \pm i \left[ \omega_o t + (k-\xi_0). \vec{r} \right]
\]

In eq. (28), \( a_k^+ \) and \( a_k^- \) are the creation and annihilation operators, and \( N(k) \) is the vacuum mode density:

\[
N(k) = \sqrt{\frac{\hbar}{2 e_o \omega. (2\pi)^3}}
\]

In eq. (29), \( r_i^+ \) and \( r_i^- \) are the \( i \)th atom rising and lowering operators:
Note that the semi-classical polarization (eqs. 2 and 3) is nothing but the expectation value of the polarization operator (eq. 29).

Let us now study the superradiant emission build-up. During this process, the inversion density \( \mathcal{N} \) keeps its initial value \( \mathcal{N}(t=0) \) and Bloch-Maxwell's set of equations is linear. It is thus possible to consider the formal solutions for the field \( \xi^+ \):

\[
\xi^+ = \int_{-\infty}^{t} dt' \xi^+(z=0,t'). G_1(t') + \int_{0}^{z} dz' P^+(z',t=0). G_2(z')
\]  
(33)

where \( G_1 \) and \( G_2 \) are Bloch-Maxwell's Green functions associated to the boundary condition \( \xi^+(z=0,t) = \delta(t,t') \) and \( P^+(z,t=0) = \delta(z,z') \).

The normal order expected emitted intensity \( < I_N > \) is then:

\[
< I_N > = 2 < \xi^+, \xi^- > = 2 \int_{0}^{\infty} dt' \int_{0}^{\infty} dt \xi^+(t') \xi^-(t',z=0) 
\]  
(33)

Equation (33) means that \( < I_N > \) is only sensitive to the incoming boundary field \( \xi^+(t,z=0) \) [18]. Expansion of \( < \xi^+ \xi^- > \) using eq. (28) shows that the initial z=0 noise intensity is proportional to \( a_k^- a_k^+ \) i.e. \( n_\omega + 1 \). The \( n_\omega \) factor can be attributed to the blackbody field noise and the unity factor to spontaneous emission noise. Numerical simulations can thus be performed with a z=0 random field given by eqs (24) and (25) with a different (from the pure blackbody case of eq. (26)) mean square value \( \xi_0 \):

\[
(n_\omega + 1) \frac{\hbar \omega}{2} = \frac{1}{2} \epsilon_0 \Delta x \Delta y \cdot c \Delta t \xi_0^2 
\]  
(34)

Note that eq. (34) remains valid even at zero temperature (\( n_\omega = 0 \)) and describes then the pure spontaneous emission noise.

A similar study can be performed for the antinormal order expected intensity \( < I_A > \). One gets:
\[ \langle I_A \rangle = 2 \langle \xi^- \xi^+ \rangle = \]
\[ = 2 \int dt' | C_1(t') |^2 \langle \xi^- \xi^+ \rangle_{t'=0, z=0} + \]
\[ + 2 \int dz | C_1(z') |^2 \langle \xi^- \xi^+ \rangle_{z=0} \]
\[ = 2 \int dz | C_1(z') |^2 \langle \xi^- \xi^+ \rangle_{z=0, t=0} \]
\[ \tag{35} \]

In eq. (34), the first term involving \( \langle \xi^- \xi^+ (t', z=0) \rangle \), i.e. \( a_k a_k^* \) or \( n_\Psi \), describes the blackbody field. The second term with \( \langle \xi^- \xi^+ (t=0, z) \rangle \) is related to spontaneous emission. Numerical simulations can be made using eqs. (24), (25) and (26) for the \( z=0 \) blackbody random field \( \xi^+ \). The \( \langle \xi^- \xi^+ \rangle_{t=0, z} \) term yields to a Gaussian random polarization \( \xi^+ (t=0) \). More precisely, one gets for \( \xi^+ (t=0) \) where \( \xi^+ \) is the random space average over the \( \Delta x. \Delta y. \Delta z \) boxes corresponding to the numerical grid:

\[ \overline{\xi^+} (t=0) = \text{exp} i \Psi \]
\[ \tag{36} \]

Here \( \Psi \) is an uniformly distributed random phase, and \( |\overline{\xi^+}| \) is a random amplitude whose probability \( p (|\overline{\xi^+}|^2) \) is:

\[ p (|\overline{\xi^+}|^2) \propto \exp (- |\overline{\xi^+}|^2 / \overline{\xi^+}) \]
\[ \tag{37} \]

\[ \overline{\xi^+} = 2d. \sqrt{N (t=0) / \Delta x. \Delta y. \Delta z} \]
\[ \tag{38} \]

As soon as a few superradiant photons have been emitted, the normal and antinormal intensities \( I_N \) and \( I_A \) are practically equal, and the normal order boundary conditions (eqs. (24), (25), (34)) are equivalent to the antinormal order ones (eqs. (24), (25), (26) and eqs. (36), (37), (38)).

Note that it is possible to compute any operator or correlation function expectation value. This is done in three steps:

1) A four-dimensional discretization is made, and a three-dimensional set of initial boundary values for \( \xi^+ (z=0) \) and/or \( \overline{\xi^+} (t=0) \) is randomly chosen, corresponding either to normally or antinormally ordered operators.
ii) A numerical experiment is performed by solving numerically Bloch-Maxwell's equations (eqs (5), (6), (7)). One gets classical possible trajectories for the atoms: \( \mathcal{N} \), \( \mathcal{J}^\pm \), and the field operators: \( \mathcal{E}^\pm \), for their product: \( I_A = \mathcal{E}^+ \mathcal{E}^- \), and for any correlation function: \( \mathcal{E}^+(t) \mathcal{E}^-(t-t) \ldots \)

iii) A large number of numerical experiments are performed \([8]\), and an average is made on the classical trajectories corresponding to the operator or to the correlation function studied. This gives the corresponding expectation value.

5. Dephasin g processes: Superradiance and amplified spontaneous emission

In this section, we will see how superradiant emission is modified by dephasing processes, as Doppler effect \([17]\). This effect can be rigourously described by introducing velocity varying \( \mathcal{N}(v) \) and \( \mathcal{J}^+(v) \) envelopes \([6]\).

Here, to make the study easier, we will consider a very simple, but physically meaningful model.

i) The inversion density \( \mathcal{N} \) is supposed to be a known quantity, and Bloch's equation (6) is not considered. This assumption is valid, for describing the build-up of the superradiant emission as in section 4. It is also valid in any situation where the atomic inversion density is not governed by the superradiant emission itself, but by other phenomena: emission on other transitions, plasma kinetics...

ii) The studied dephasing process is simply described phenomenologically by adding a \((-1/T)\mathcal{J}^+\) atomic polarization damping term on the right hand side of Bloch's equation (\( T \) is here the polarization damping time).

iii) Transverse effects are neglected.

The evolution is then described by the equations:

\[
\begin{align*}
\frac{\partial}{\partial z} \mathcal{E}^+ &= \frac{i\omega}{2\varepsilon_0 c} \cdot \mathcal{J}^- \\
\frac{\partial}{\partial t} \mathcal{J}^+ &= \frac{1}{T} \cdot \mathcal{J}^+ = \frac{2id^2}{\hbar} \mathcal{E}^- \cdot \mathcal{N}
\end{align*}
\]
At the very beginning of the emission, $\mathcal{T}^+\!$ is very small, and the $(1/T)\mathcal{T}^+$ dephasing term on the left-hand side of eq. (40) is neglible. The superradiant emission is thus able to build-up and to reach a steady state where the damping $(1/T)\mathcal{T}^+$ term is equal to the eq. (40) right-hand side source term. The emission build-up corresponds to superradiance $(1/T)\mathcal{T}^+\! = 0$, the steady state $(\partial/\partial t)\mathcal{T}^+\! = 0$ to amplified spontaneous emission (A.S.E.), the real evolution passing continuously from one (S.R.) to the other (A.S.E.).

The superradiant solution $(1/T)\mathcal{T}^+\! = 0$ obeys to:

$$\begin{align}
\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \begin{cases} 
\mathcal{T}^+ \! = \mathcal{L}_{\mathcal{T}^+} \mathcal{P}^+ \! \\
\mathcal{C}^+ \! = \frac{1}{L \cdot T_R} \mathcal{C}^+ \! \end{cases} \end{align} \tag{41}$$

where $T_R$ is the superradiant time:

$$\frac{1}{T_R} = \frac{3}{8\pi} \Gamma_{SP} \cdot \mathcal{L} \cdot \lambda^2 \cdot L \tag{42}$$

In the case of a light amplifier $\mathcal{E}^+(z=0,t) = \mathcal{C}_0$, one gets an analytical solution for $\mathcal{C}^+$ (modified Bessel function) $[6]$

$$\mathcal{C}^+ = \mathcal{C}_0 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left[ \frac{z \cdot t}{L \cdot T_R} \right]^n \tag{43}$$

Similarly, the A.S.E. solution $(\partial/\partial t)\mathcal{P}^+\! = 0$ obeys to:

$$\frac{\partial}{\partial z} \mathcal{C}^+ \! = \frac{1}{L} \cdot \frac{T}{T_R} \cdot \mathcal{C}^+ \! \tag{44}$$

One gets for the light amplifier:

$$\mathcal{C}^+ = \mathcal{C}_0 \cdot \exp(\alpha z) \tag{45}$$

where:

$$\alpha = T \cdot (L \cdot T_R) \tag{46}$$

is simply the amplitude linear gain of active medium.

The superradiant solution is valid, at the very beginning of the emission, when the S.R. solution (eq.43) is smaller than the A.S.E. steady state (eq.45). This occurs for $t < \eta \cdot T$, where $\eta$ is a numerical
factor which varies slowly with the gain of the active medium: \( \eta = 2.24 \) and 7.09 for amplitude gain \( \exp(T/T_R) \) respectively equal to \( 10^2 \) and \( 10^{10} \).

In more complex situations (where assumptions i), ii), iii) are not valid), the emission build-up still takes place in the superradiant regime. A time \( \eta \cdot T \) later amplified spontaneous emission begins. Exact value of \( \eta \) is no more given by eqs (43) and (45), but as \( \eta \) varies very slowly with the gain, \( \eta \) remains of the order of 2 to 7. Superradiance is characterized by the superradiant time \( T_R \) (eq. 42). A.S.E. is characterized by the amplitude linear gain \( \alpha \). \( \alpha \) and \( T_R \) are related together by equation (46).

**Conclusion**

Bloch–Maxwell's equations are well suited for describing electromagnetic emission of collective system of atoms. There are able both to deal with fundamental effects as spontaneous emission, and to fit with the real experimental situation, including any physical effect. Here we have briefly discussed how to deal with diffraction (§2) and atomic dipole damping (§5).

Bloch–Maxwell’s equations have generally no analytical solution and a rather large amount of numerical computing is needed in order to simulate a given experiment. In particular, fluctuations are studied by making statistics on a large number of numerical experiments (§3, 4).

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