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THE QUASI-CRYSTALLINE QUANTUM SPIN-½ XY-CHAIN

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Résumé — On étudie une chaine de spins quantiques avec des couplages "quasicristallins" dans le plan XY et un champ magnétique uniforme le long de l'axe des Z. L'aimantation à température nulle est une fonction de Cantor du champ magnétique. La chaleur spécifique en champ nul se comporte comme $T^\Delta$ à basse température ; l'exposant $\Delta$ varie continûment entre zéro et un.

Abstract — The quantum spin-½ chain with "quasicrystalline" couplings in the XY-plane and a steady magnetic field in the Z-direction is studied. The zero-temperature magnetization is a Cantor-function. The zero-field specific heat behaves at $T^\Delta$ at low temperature with $\Delta$ varying continuously between zero and one.

INTRODUCTION

Recently several one-dimensional quasicrystalline systems have been studied. Luck and Petritis [1] considered a quasicrystalline phonon model, which is presented in shorter form in this issue [2]. Then Luck considered a quasicrystalline Ising model at zero temperature [3]. The present contribution summarizes a recent work done in collaboration with Luck [4].

We consider the Fibonacci tiling of the line, explained for instance in Ref. [2] (this issue). To long bonds we attach a coupling strength $J_1 = \rho$, and to short bonds a coupling $J_1 = 1$. Given this sequence $\{J_1\}$, we consider the quantum spin-½ XY Hamiltonian

$$H = 2 \sum_{i=0}^{N} J_1 \left( S_{i}^{x} S_{i+1}^{x} + S_{i}^{y} S_{i+1}^{y} \right) - \hbar \sum_{i=0}^{N} S_{i}^{z}$$

(1)

where $\hbar$ represents a magnetic field in the Z-direction. The $S_{i}^{\alpha}$ are half the Pauli matrices. They obey:

$$[S_{i}^{x}, S_{j}^{\beta}] = i \delta_{ij} \varepsilon^{\alpha\beta\gamma} S_{i}^{\gamma} ; \sum_{\alpha} S_{i}^{\alpha} S_{i}^{\alpha} = \frac{3}{4}$$

(2)

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The reason why the model (1) can be solved is that it can be reduced to a free fermion problem. Lieb et al [5] showed this for the "pure" case \((J_i = J)\) and Smith [6] noted that it can be done as well in the case of arbitrary \(J_i\). The result is that the free energy has the form

\[
-\beta f = -\frac{1}{2} \beta n + \int_{-\infty}^{\infty} \ln \left(1+e^{\beta(E-h)} \right) dH(E)
\]

(3)

where \(H(E)\) is the integrated density of states

\[
H(E) = \lim_{N\to\infty} \frac{1}{N} \left( \# \ E_\alpha < E \right)
\]

(4)

of the tight-binding problem

\[
J_i u_{i-1} + J_{i+1} u_{i+1} = E_\alpha u_i \quad (u_{-1} = u_N = 0)
\]

(5)

Since \(H(E)\) will turn out to be a Cantor function, one cannot introduce a density of states \(\rho(E)\) by \(dH(E) = \rho(E) dE\) in Eq.(3). If \(E_\alpha\) is an eigenvalue of Eq.(5), so is \((-E_\alpha)\), as is seen by substituting \(u_i = (-1)^i v_i\). Hence \(H(E) = 1-H(-E)\); \(H(0) = \frac{1}{2}\) and we only have to consider positive energies.

The spectrum of Eq.(5) can be studied most conveniently by introducing \(Q_n = J_0 J_1 ... J_n u_n\). Its equation of motion can be written in the transfer matrix form

\[
\begin{pmatrix}
Q_{n+1} \\
Q_n
\end{pmatrix} =
\begin{pmatrix}
E & -J_n^2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
Q_{n+1} \\
Q_n
\end{pmatrix} = T_n \begin{pmatrix}
Q_0 \\
Q_{-1}
\end{pmatrix}
\]

(6)

where \(T_n\) is a product of \((n+1)\) 2x2 matrices. See also Ref.[2] for a closely related situation. For the Fibonacci chain obtained by projection and discussed in [2], the \(T_{F_L}\) satisfy a specific recurrence relation. Their normalized traces \(x_L = \frac{1}{2} \text{tr} \ (T_{F_L})\) and determinants \(y_L = \det(T_{F_L})\) satisfy the recurrence relations derived by Kohmoto et al [7] and Luck [3]:

\[
x_L = 2x_{L-1}x_{L-2} - y_{L-2}x_{L-3}
\]

(7)

\[
y_L = y_{L-1}y_{L-2}
\]

we find \(y_0 = 1\), \(y_1 = \rho^2\) and hence \(y_L = \rho^{2F_{L-1}},\) where \(F_L\) are the Fibonacci numbers.

This leads us to define \(E_L = \rho^{-F_{L+1}}x_L\), which satisfies

\[
E_L = 2E_{L-1}E_{L-2} - E_{L-3}
\]

(8)
The boundary conditions are most easily expressed as

\[ \xi_{-2} = \frac{1}{2} (\rho + \rho^{-1}), \quad \xi_{-1} = \frac{E}{2}, \quad \xi_0 = \frac{E}{2\rho} \]  

(9)

The map (8) has a conserved quantity [7]

\[ I = \xi_{L-2}^2 + \xi_{L-1}^2 + \xi_L^2 - 2\xi_{L-2}\xi_{L-1}\xi_L = \frac{1}{4} (\rho + \rho^{-1})^2 \]  

(10)

**Fig.1:** Plot of the integrated density of states \( H(E) \) against energy \( E \), for \( \rho=2 \).

**The Spectrum**

In Fig.1 we present a plot of the integrated density of states \( H(E) \) for the value \( \rho=2 \). It is seen that \( H \) is a Cantor function. Its behavior near the maximal energy \( E_{\text{max}} \) is

\[ 1 - H(E_{\text{max}} - \xi) \sim \xi^\Delta P \left( \frac{\ln \xi}{\ln |\lambda_2|} \right) \]  

(11)

where \( P \) is a periodic function with unit period. The exponent \( \Delta \) and the scale \( |\lambda_2| \) follow from an argument given in [1]: The map (8) has a six-cycle \( \alpha \to -\beta \to -\alpha \to \beta \to -\alpha \to -\beta \). The largest eigenvalue (in absolute value) is \( \lambda_2 \) with

\[ \lambda_2 = -\left( \frac{1}{2} + \sqrt{\frac{S}{2} - \frac{3}{4}} \right) \]  

(12)
A scaling argument then yields $\Delta = 2 \ln r / \ln |\lambda_2|$. The very same behavior is present near gap edges $E_g$ inside the spectrum. In particular $\Delta$ does not depend on $E_g$; the periodic function $P$ may do so, however.

At $E=0$ one finds that the initial conditions (9) already lie on a six-cycle $\frac{\mu}{2} \to 0 \to 0 \to -\frac{\mu}{2} \to 0 \to 0$ discussed by Kohmoto et al [7]. Hence one has a similar result

$$H(E) \sim E^\Delta P_0 \left( \frac{\ln E}{\ln \lambda_3} \right)$$

(14)

where the largest eigenvalue of the linearized map is $\lambda_3^2$ with

$$\lambda_3 = \frac{1}{2} \left[ \mu^2 + (4 + \mu^4)^{1/2} \right]; \quad \mu = \rho + \rho^{-1}$$

(15)

and where $\bar{\Delta} = 3 \ln \tau / \ln \lambda_3$.

**THERMODYNAMICS**

From Eq.(3) one notes that the external field $h$ plays the role of the Fermi level. At $T=0$ all states with $E \leq h$ are filled up. One finds, for instance, for the magnetization

$$M(h,T=0) = - \frac{\partial F}{\partial h} \bigg|_{T=0} = H(h) - \frac{\Delta}{2}$$

(16)

Hence $M$ is a Cantor function, with the scaling behavior (11) at its gap edges $E_g \neq 0$ and the behavior (14) at $h = 0$. The zero temperature susceptibility is an ill-defined quantity (zero or infinite).

From (3) and (14), we deduce for the specific heat

$$C(h=0;T) \sim \frac{\ln T}{\ln \lambda_3} R_0 \left( \frac{\ln T}{\ln \lambda_3} \right)$$

(17)

where $R_0$ is a periodic function with unit period, expressible in terms of $P_0$.

Since $0 < \bar{\Delta} < 1$, Eq.(17) interpolates between the uniform case ($\rho=1; \bar{\Delta}=1$) and the totally random case ($C \sim (\ln T)^{-2} \Rightarrow \bar{\Delta} = 0^*$).

The susceptibility behaves as

$$\chi(h=0;T) \sim \frac{\ln T}{\ln \lambda_3} R_1 \left( \frac{\ln T}{\ln \lambda_3} \right)$$

(18)
Plots of $C(h=0;T)$ and $\chi(h=0;T)$ are presented in Figs. 2 and 3 for the case $\rho=3$. It is seen that $C$ may exhibit an infinity of regions of non-monotonic behavior.

Fig. 2: Log-Log plot of the zero-field specific heat against temperature, for $\rho=3$.

For $h = E_g$, relations similar to Eqs. (17)-(18) hold, with $\Delta$ and $|\lambda_2|$ replacing $\bar{\Delta}$ and $\lambda_3$, respectively. For values of $h$ inside one of the gaps, low-temperature excitations are damped exponentially: nothing interesting happens in a desert.

Fig. 3: Same as Fig. 2 for the zero-field susceptibility.
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