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RENORMALISATION ON THE PENROSE LATTICE

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Introduction and summary

The discovery of quasicrystals has started an intense theoretical activity. Much of this activity is devoted to the study of the structure and crystallography of these new phases. Other studies concern their stability, their elementary excitations...

We investigate here the magnetic structure of quasicrystals: we want to study the critical behaviour of the Ising model with Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j$$

(1)

where $\langle i,j \rangle$ are nearest neighbour sites on a Penrose tiling. The variables $S_i=\pm 1$ are Ising spins. The Penrose tiling will be described here using Robinson tiles [1,2].

In this talk we will mainly explain how, by taking advantage of the self-similarity of the Penrose tiling, it is possible to match a real space renormalisation method to the deflation (composition) of the tiling. This will be illustrated here by the cumulant method [3] applied to the dual of the Penrose tiling. We will then give a brief account of the results. The remainder of this section is a summary of the talk.

There are two kinds of Robinson triangles called P and Q (Fig.1). One may associate to these two triangles two kinds of atoms with spins located at the center of the triangles, i.e. on the dual lattice. The magnetic interactions between the atoms are described by coupling constants $J_{ij}$. 
Figure 1. Robinson triangles and the process of composition. Angles are in units of $\pi/5$. $\tau = (\sqrt{5} + 1)/2$

which depend on the environments of the triangles. One has thus different types of couplings between atoms $P$ and $Q$, $P$ and $Q$, $P$ and $Q$.

The renormalisation method consists in renormalising in two steps, using two types of blocks. The cumulant method will be performed to first order only. We find ferromagnetic and antiferromagnetic phase transitions corresponding to the same fixed point. Another renormalisation scheme, namely the Migdal-Kadanoff approximation, applied on the Penrose tiling itself, and an extensive study of the antiferromagnetic phases will be described with full details elsewhere [4].

Penrose tiling with Robinson triangles

The Penrose tiling is usually described with darts and kites or with two rhombs. If one cuts those polygons in a particular way one obtains the two
triangles $P$ and $Q$ of Fig. 1. They will be referred to as A-tiles. The tiling with A-tiles is subject to the following two matching rules.

1. Each edge of a triangle must be abutted by the edge of another triangle in such a way that the colors of the vertices match.

2. In the case of a monochromatic edge (i.e. an edge joining 2 vertices of the same color), the smaller angle of one triangle must abut the smaller angle of the other. This is equivalent to say that each monochromatic edge is oriented. The monochromatic edges of adjacent tiles must have the same orientation.

The A-tiles can be composed to form new tiles (Fig. 1). Composing $P$ and $Q$ gives the triangle $tQ'$ which may be obtained from $Q$ by expanding by the factor $t$ and reversing the colors. Thus there are two basic cases of tiling with triangles, A and B. In case A, there is a large triangle $P$ and a small triangle $Q$. In case B, the same triangle $P$ has become the small triangle, and the large triangle is $tQ'$. Similarly the composition of $tQ'$ and $P$ form the triangle $tP'$. Consequently there is a third case of tiling denoted $tA'$ consisting of triangles $tP'$ and $tQ'$. This is isomorphic to case A, since both triangles are expanded by the factor $t$ and have the colors reversed (Robinson [1]). We may proceed further: by composition, the $tA'$-tiles lead to $tB'$-tiles, the $tB'$-tiles to $t^2A$-tiles. Let us summarise the successive steps (Fig. 2):

In the parentheses we wrote in the first place the larger tile that appears in the tiling. The important remark for what follows is that at each step in the composition process, the links or edges that are deleted are all contained in a particular polygon (Fig. 2 and 3). In the first and third steps the compositions of tiles take place in a rhomb. In the second and fourth steps they take place in a kite.
Figure 2. From the A-tiling to the $\tau'$-A-tiling. Hatched polygons are rhombs in a) and c), kites in b) and d). (This figure is a modified version of a figure of Ref. [2])

- $P + Q \rightarrow \tau Q'$
- $A(P, Q) \rightarrow B(\tau Q', P)$
- $\tau Q' + P \rightarrow \tau P'$
- $B(\tau Q', P) \rightarrow \tau A'(\tau P', \tau Q')$
- $\tau P' + \tau Q' \rightarrow \tau^2 Q$
- $\tau A'(\tau P', \tau Q') \rightarrow \tau B'(\tau^2 Q, \tau P')$
- $\tau^2 Q + \tau P' \rightarrow \tau^2 P$
- $\tau B'(\tau^2 Q, \tau P') \rightarrow \tau^2 A(\tau^2 P, \tau^2 Q)$
The dual lattice. Renormalisation in two steps

The method we use consists in renormalising the Hamiltonian in two steps. The first step corresponds to the passage from the A-tiling to the B-tiling. The second step to the passage from the B-tiling to the \( \tau A' \)-tiling. In the first step the block chosen is the rhomb, in the second step the block is the kite. Since the resulting lattice is isomorphic to the initial one with inverted colors on the vertices, we do not consider the two next steps leading to the \( \tau^2 A \)-tiling.

Furthermore, in order to be able to implement a block spin renormalisation method—such as the cumulant method—one has to define block spins, and separate the bonds into intra-block bonds and inter-block bonds. This separation is easily done if we locate the spins on the dual lattice instead of the direct one. Thus one has to consider the environments around each

Figure 3. The process of composition (deflation).
triangle at each step (Fig. 4). These environments define 8 coupling constants \( K_1 - K_8 \) at the first step and 5 coupling constants \( L_1 - L_5 \) at the second step (those coupling constants are related to the \( J_{ij} \) of Eq.(1) by including the temperature in their definition: \( K_\alpha = J_{ij} / kT \)).

The first renormalisation leads to the relation

\[
(L_1, L_2, L_3, L_4, L_5, K_6, K_7, K_8) = R_1(K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8)
\]

The second one to the relation

\[
(K'_1, K'_2, K'_3, K'_4, K'_5, K'_6, K'_7, K'_8) = R_2(L_1, L_2, L_3, L_4, L_5, K_6, K_7, K_8)
\]

Finally one has

\[
(K'_1, K'_2, K'_3, K'_4, K'_5, K'_6, K'_7, K'_8) = R_2 \cdot R_1(K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8)
\]

After those two steps the resulting lattice is scaled by \( \tau \). The \( R_1 \) and \( R_2 \) renormalisation transformations are given at first order with the cumulant method by the following equations.

**Renormalisation equations**

Defining \( x = 1/(1+\exp(-2K_\alpha)) \) and \( y = 1/(1+\exp(-2L_\alpha)) \) the renormalisation equations read:

1) At step \( R_1 \)

\[
\begin{align*}
L_1 &= x^2 (K_3 + K_7) \\
L_2 &= x K_6 \\
L_3 &= x^2 K_6 \\
L_4 &= x^2 K_1 \\
L_5 &= x K_2
\end{align*}
\]

2) At step \( R_2 \)

\[
\begin{align*}
K'_1 &= L_1 \\
K'_2 &= y L_1 \\
K'_3 &= L_3 \\
K'_4 &= L_4 \\
K'_5 &= y L_5 \\
K'_6 &= y^2 (L_4 + K_8) \\
K'_7 &= y^2 L_1 \\
K'_8 &= y^2 K_7
\end{align*}
\]

These equations admit a unique fixed point: \( x^* = .82661, y^* = .88215 \). One finds a ferromagnetic transition: the isotropic critical point is given by \( K_1 = K_c = .959970 \). When certain couplings are negative, one finds antiferromagnetic ordering at low temperature, corresponding to the same ferromagnetic fixed point. The complete phase diagram is difficult to
Figure 4. Different types of coupling constants associated to different environments, for the A-tiling (P and Q triangles) and for the B-tiling (\(\tau Q'\) and P triangles).

describe since the space of coupling constants is of dimension eight. This is nevertheless done in Ref. [4] for the case of the Penrose tiling itself. This case is more tractable since one has four coupling constants to describe the system, and one finds an infinite number of antiferromagnetic phases.

References