USE OF VARIATIONAL EQUATIONS TO ANALYZE EQUILIBRIUM AND STABILITY OF AN ELECTROSTATICALLY STRESSED CONDUCTING FLUID: APPLICATION TO A CUSPIDAL MODEL OF AN LMIS

M. Chung, P. Cutler, T. Feuchtwang, E. Kazes, N. Miskovsky

To cite this version:

HAL Id: jpa-00224406
https://hal.archives-ouvertes.fr/jpa-00224406
Submitted on 1 Jan 1984
USE OF VARIATIONAL EQUATIONS TO ANALYZE EQUILIBRIUM AND STABILITY OF AN ELECTROSTATICALLY STRESSED CONDUCTING FLUID: APPLICATION TO A CUSPIDAL MODEL OF AN LMIS+

M. Chung, P.H. Cutler, T.E. Feuchtwang, E. Kazes and N.M. Miskovsky++

Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Abstract - The set of variational equations, recently derived by Sujatha et al., for the equilibrium configuration of a conducting fluid in an electric field are applied to the cuspidal model of a liquid metal ion source. It is found that the mathematical singularity at the apex of an ideal cusp is not compatible with any static equilibrium configuration. To circumvent some inherent difficulties in the use of the variational formalism, a virtual-work-energy analysis is used to derive new equations for the study of equilibrium and stability of stressed conducting fluids. Comparison with the corresponding Taylor equilibrium and stability conditions reveals important differences.

I. INTRODUCTION

Recently, Sujatha et al. /1,2/ have used the variational formalism to derive an Euler-Lagrange Equation, with associated boundary conditions, for the energetically allowed equilibrium states, i.e., configurations of an electrically stressed conducting fluid. This variational analysis is based on the minimization of the total energy of the stressed fluid in an electric field, subject to appropriate constraints on the physical system. In the problem we have considered the constrained quantity is the volume of the fluid, which, however, can be arbitrarily small or as large as we wish consistent with the requirements of the physical problem.

The relevant equations to be used in the present analysis consist of the Euler-Lagrange equation and four associated boundary conditions given by Eqs. (12) and (11) respectively in Reference 1. In addition, a fifth condition, given by Eq. (24) in Reference 2, is included which determines "the free end point" of the fluid in the model described by Sujatha et al. We shall refer to this complete set of equations as the Euler-Lagrange (EL) equations.

The validity of these equations has been verified for the case of a concentric fluid source and extractor electrode held at constant potential by an external power supply. Although this is a simple example, the concentric sphere model satisfies all the physical requirements of an electrostatically stressed conducting fluid in equilibrium.
Furthermore it has been shown that no rigid or ideal fluid cone (e.g., the so-called Taylor cone /1,3/) can satisfy the EL equations. That is, an electrically stressed conducting liquid cannot start from the zero field condition and pass quasi-statically through a series of energetically allowed equilibrium states and form a cone with half-angle of 49.3°—or for that matter of any other angle. That such a process is not allowed energetically is a consequence of the energy minimization principle inherent in the variational principle. Although Taylor’s electrostatic solution for his ideal conical model, that is, the conducting cone and curved counter-electrode /3,4/, satisfies his stress condition, which requires that the net force vanish, this is only the necessary but not sufficient condition for the existence of a stable structure. The sufficient condition is that a stressed fluid must assume a shape consistent with both energy equilibrium of forces and minimization.

II. APPLICATION OF THE CUSPIDAL MODEL OF AN LMIS

We have used the set of variational equations discussed above to determine if, and under what conditions, the cuspidal shape, observed in the dynamic state of an LMIS, by Gaubi et al. /5/ and others /6/, might also be the precursor equilibrium shape of the stressed fluid in an LMIS prior to the onset of instabilities. The verification of the proposed cuspidal shape requires the solution of Laplace’s equation for the potentials and associated fields and charge densities for an ideal spherical cusp and planar counter-electrode model of an LMIS. The exact solution of Laplace’s equation for this model, subject to the boundary conditions, \( V = V_0 \) on the cusp and \( V = 0 \) on the planar electrode, has been obtained by Miskovsky et al. /7/ using bi-spherical coordinates \((\mu, \eta, \phi)\). The notation used here is defined in Reference 7. The potential is given by /7/:

\[
V(\mu, \eta) = \frac{2V_0}{\eta} \sqrt{\cosh\mu - \cos\eta} \int_0^\infty d\tau \int_0^{\pi/2} d\beta \frac{\sin\tau \sin^2\beta}{\cosh\mu - \cos\eta} \frac{P_{1/2 + i\tau}(-\cos\eta)}{P_{1/2 - i\tau}(-\cos\eta)}
\]

An equivalent form for \( V(\mu, \eta) \), which will be used in the following discussion is /8/:

\[
V = V_0 + \sqrt{\cosh\mu - \cos\eta} \sum_{\nu_S} A_{\nu_S} e^{-(\nu_S + 1/2)} P_{\nu_S}(-\cos\eta)
\]

where \( A_{\nu_S} = \frac{V_0}{H_{\nu_S}(-\cos\eta)} \int_{-\cos\eta_0}^{1} P_{\nu_S}(-\cos\eta) d(-\cos\eta) \)

and \( H_{\nu_S}(-\cos\eta_0) = \int_{-\cos\eta_0}^{1} [P_{\nu_S}(-\cos\eta)]^2 d(-\cos\eta) \).

The values of \( \nu_S \) are determined from the relation

\[
P_{\nu_S}(-\cos\eta_0) = 0.
\]

On the surface of the cusp, defined by \( \eta = \eta_0 \), the electric field is given by

\[
E_\eta = -\frac{1}{h_\eta} \left. \frac{2V_0}{3\eta} \right|_{\eta_0} = -\frac{\cosh\mu - \cos\eta_0}{a} \left( \frac{3V_0}{3\eta \eta_0} \right)
\]

where \( h_\eta \) is the metric /8/ and ‘a’ is the apex-counter-electrode spacing. Using Eqs. (2) - (5), \( E_\eta \) can be written as

\[
E_\eta(\mu) = -\frac{1}{\sin^2 \eta_0} \sum_{\nu_S} A_{\nu_S} e^{-(\nu_S + 1/2)\mu} (\cosh\mu - \cos\eta_0)^{3/2}
\]

with \( A_{\nu_S} = \nu_S P_{\nu_S-1/2}(-\cos\eta_0) A_{\nu_S} \).
To express the field in terms of the cylindrical coordinates \( z \) and \( \rho \) we use
\[
z = \frac{\sinh \eta}{\cosh \eta - \cos \eta}
\]
and the equation of the cusp,
\[
(r - \cot \eta)^2 + z^2 = a^2 \csc^2 \eta_0
\]
Then, \( E(\rho) = E(\rho, z) = \sum \frac{A_\nu S}{4\pi} (1 - \frac{z}{a} + \frac{\rho}{a} \cot \eta_0)^{\nu S + 1/2} \left( \frac{\sin \eta_0}{\rho} \right)^{\nu S + 2} \)
and the surface charge density, \( \sigma \), on the cusp is
\[
\sigma(\rho) = \frac{1}{4\pi} \frac{\tau \cdot E}{E_\eta} = \frac{1}{4\pi} \frac{\sum \frac{A_\nu S}{4\pi} \left(1 - \frac{z}{a} + \frac{\rho}{a} \cot \eta_0\right)^{\nu S + 1/2}}{\left( \frac{\sin \eta_0}{\rho} \right)^{\nu S + 2}}
\]
After some manipulation, the charge density can be written in the following form:
\[
\sigma = \sigma(\rho) = \sum \frac{C_\nu S}{\nu S} \left( \frac{\sin \eta_0}{\rho} \right)^{\nu S + 2} \left(1 - \frac{z}{a} + \frac{\rho}{a} \cot \eta_0\right)^{\nu S + 1/2}
\]
where \( C_\nu S = \frac{\nu S}{4\pi \sin \eta_0} \).

It can be shown that the total charge over any region of the cusp including the apex is finite. The consistency of the cuspidal model implies that one can find a value of \( \eta_0 \) such that Eq. (13), for \( \sigma \), will satisfy the variational equation /1/. For convenience we express that equation in the following form:
\[
\text{LHS} \equiv 2 \left[ (\gamma - \frac{V_0}{2} \sigma) \frac{1}{R_1} + \frac{1}{R_2} + \lambda \right] = F(\rho(z), \rho'(z), \rho''(z)) \equiv \text{RHS}
\]
where \( \rho = \rho(z) \) is the equation of the cuspidal surface and \( \rho' \equiv \partial \rho/\partial z \), etc. (See Eq. (10)). The surface tension is denoted by \( \gamma \) and \( \lambda \) is the Lagrangian multiplier. Using the following relationship for twice the mean radius of curvature for the cusp
\[
\frac{1}{R_1} + \frac{1}{R_2} = \cot \eta_0 + \frac{2}{acsc \eta_0}
\]
and Eq. (13) for \( \sigma \), it can be shown that the LHS of Eq. (14) has the form \( \text{Const} + A/\rho + \text{(non-integer power series in } \rho \text{)} \) whereas the RHS can be written as \( \text{Const}' + \Sigma' \text{(non-integer power series in } \rho \text{)} \) where prime denotes constants and coefficients different from the unprimed values. It is also found that, in general, for arbitrary values of \( \eta_0 \), the RHS does not contain the inverse power \( \rho^{-1} \). Thus it appears that Eq. (14) is not satisfied. However, it is important to note that the determination of the non-integral indices \( \nu S \) requires a numerical solution of the Legendre functions of non-integral index for which there are only incomplete tabulations. Therefore, the determination of the existence of the inverse power of \( \rho \) on the RHS, for an arbitrary value of \( \eta_0 \), which requires the summation of a complicated infinite series, does not yield an unequivocal result.

In order to understand the significance of the term proportional to \( \rho^{-1} \), we note that this is the leading term in the expansion of \( \sigma \) in powers of \( (\rho/a) \) close to the apex, where \( \rho/a \ll 1 \):
\[
\sigma(\rho) \approx \sum \frac{C_\nu S}{\nu S} \frac{1}{2^\nu S + 1/2} \left( \frac{\rho}{a \sin \eta_0} \right)^{\nu S - 1} \equiv \sum \frac{C_\nu S}{\nu S} \left( \frac{\rho}{a \sin \eta_0} \right)^{\nu S - 1}
\]
In this case the variational equation reduces to

\[
\frac{2}{V_0} \left( \lambda - \frac{2\gamma}{\text{as} \theta_0} \right) + \frac{2\gamma \cos \theta_0}{V_0} \frac{1}{\rho} + \sum \left( \frac{\nu_{s+1}}{\text{as} \theta_0} + \frac{\nu_s \cos \theta_0}{\rho} \right) B \nu_s \left( \frac{\rho}{\text{as} \theta_0} \right) \nu_{s-1} = 0 . \tag{17}
\]

Again the validity of Eq. (17) cannot be established unequivocally because it has not been possible to prove that the coefficients of each of the non-integer powers of \( \rho \) is identically zero. The preceding suggests the conjecture that the reason for the above difficulty is the mathematical singularity at the apex of the ideal cusp which is not compatible with a static equilibrium shape. That is, the singularity leads to infinitely large fields at the apex. To obviate this difficulty, one can modify the shape in the vicinity of the apex and use, for example, a three-dimensional verseliria /9/, a shape which is cusp-like, but with continuous derivatives at the apex. Such a modified geometry is consistent with the general theoretical description of stressed fluids in the capillary wave model just prior to onset of hydrodynamic instability /10,11/.

### III. DERIVATION OF THE NEW CONDITIONS FOR EQUILIBRIUM AND STABILITY

The variational formalism provides a set of equations which, in principle, determines the static equilibrium configuration of a stressed conducting fluid. However this technique is less useful in analyzing the stability of the structure. As discussed in the Introduction the first variation provides only a necessary but not sufficient condition for the existence of an extremum /12/. The second variation which is necessary to establish the sufficiency conditions leads to equations which are, in general, intractable. We have therefore used the direct virtual-work-energy analysis to investigate the condition for instability of an electrically stressed conducting fluid. This analysis yields

1) an equilibrium equation which is identified as the Laplace stress condition including electric fields, and

2) an equation describing the stability of the stressed fluid and from which it is possible to obtain a value of the critical or breakdown voltage \( V_b \).

Since the detailed derivation is given elsewhere /13/, we will only present a brief summary of the relevant analysis. Consider a fluid in equilibrium under the action of surface tension \( \gamma \) and external electrostatic forces, defined by the potential function \( V \). Assuming a displacement \( \delta \zeta \) along the normal to the fluid interface, with the potential maintained at a constant value \( V_0 \) by an external power supply, it can be shown /13/ that the work done to displace an infinitesimal element of surface area is given by \( \mathcal{F} \delta \zeta \), where \( \mathcal{F} \) is defined as the average normal force on the element of surface \( df \) during the displacement \( \delta \zeta \). Arguing by analogy with the simple dynamical problem of forced oscillatory motion, the average force can be expressed as

\[
\mathcal{F} = A + B \delta \zeta . \tag{18}
\]

If the liquid surface is initially in equilibrium, then \( \delta \zeta = 0 \) and \( \mathcal{F} = 0 \). This implies \( A = 0 \) /13/, so that

\[
A = (-p_1 + p_2) + \left( Y - \frac{1}{2} V_0 \frac{\rho}{R_1} + \frac{1}{2} V_0 \frac{\rho}{R_2} \right) - \frac{1}{2} \frac{V_0}{\partial \nu} \frac{\delta \nu}{\delta \zeta} = 0 , \tag{19}
\]

where \((-p_1 + p_2)\) is the pressure difference across interface and \( V_0 \) is the charge density on the undisplaced surface. Equation (19) describes the equilibrium condition of the undisplaced surface and is equivalent to the Laplace stress condition. If unstable oscillations are to occur, then in the force equation given by Eq. (18), it is necessary that the coefficient \( B \leq 0 \). Physically this means that if an infinitesimal force is applied to the system, the surface will not return to it's original position

\[\footnote{In the analysis we describe above, terms up to third-order in the displacement \( \delta \zeta \) are retained in order to ensure a solution valid up to second-order in \( \nu \) /14/}.
after the force is removed. In Reference 13 it is shown that for fluid surfaces of positive curvature

\[-B = \frac{1}{2} (-p_1 + p_2) + (\gamma - \frac{1}{2} V_0 g_0) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \left( 1 - \frac{R_1 R_2}{(R_1 + R_2)^2} \right) + \frac{V_0}{2} \left( \frac{d^2 c}{dt^2} \right) \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} > 0 . \]  

\[ \text{(20)} \]

IV. ANALYSIS OF THE EQUILIBRIUM CONDITION

The terms in Eq. (19) have straightforward physical interpretations.

i) The first term corresponds to the work per unit volume due to the pressure difference \((-p_1 + p_2)\) across the interface. This term is to be identified with the pressure difference due to gravitational and curvature effects as modified by an external electric field. In fact, Eq. (19) defines the pressure difference as a function of the curvature of the surface.

ii) The second term corresponds to the change in surface energy per unit volume due to an effective field-dependent surface tension. This term is also present in the EL equation, and was first discussed in Reference 1. More recently Forbes has also discussed the presence of a field dependent surface tension term obtained in his derivation of a generalized Laplace stress condition /15/.

iii) The last term corresponds to the work per unit volume needed to maintain a constant potential on the fluid, as required by the presence of an external battery.

The equation for the equilibrium condition has been verified for the concentric sphere and extractor electrode model of a stressed conducting fluid. This is simply demonstrated by noting that for a charged (fluid) sphere, radius \(R\), maintained at a constant potential \(V_0\), the pressure difference \((-p_1 + p_2)\) = 0. Since \(V_0 = E \cdot R\), \(g_0 = V_0 / 4\pi R\), so that \(1/2 V_0 (d\sigma / dt)_0 = -V_0^2 / 8\pi R^2\) and it follows that Eq. (19) is consistent with Rayleigh's criteria /3/.

We now compare this new equilibrium equation with the Taylor equilibrium condition used to describe "a conical point . . . in equilibrium" /3/:  

\[ \frac{\gamma \sigma}{R} = \frac{E^2}{8\pi} . \]  

\[ \text{(21)} \]

We note the following differences between Eqs. (19) and (21):

i) The pressure difference term is absent in Taylor's equation. Since the question of justification--or lack of it--for omitting this term has been discussed in detail elsewhere /1,2/, we here merely make the observation that even in the case of an asymptotically large system (i.e., \(R_1, R_2 \rightarrow \infty\)), Eq. (19) predicts that the pressure term \((-p_1 + p_2)\) can be "balanced" by an electrostatic contribution associated with the energy of deformation, and therefore is not necessarily equal to zero. This "balance" is not possible in the Taylor condition because the pressure term is missing; furthermore the form of Eq. (21) also requires that the field must go zero with increasing \(R\), which need not necessarily be the case e.g., an electrostatically stressed planar conducting surface with normal electric field /11/.

ii) The Taylor condition requires that the surface tension, which is independent of \(R\), balance the stress due to the electric field at every point on the surface, i.e., this condition holds globally. By contrast, the new equilibrium condition in Eq. (19) does not require that the stresses due to the surface tension and the electrostatic field be equal at all points on the surface. This is a consequence of the presence of additional position-dependent terms in Eq. (19). This is consistent with experimental observations that breakdown occurs locally /5,11,16/.
V. ANALYSIS OF THE INSTABILITY CONDITION

In Section III, we have presented the inequality (Eq. (20)) that is satisfied when a local region of the stressed fluid is in a state of unstable equilibrium. It follows therefore that any disturbance of the fluid surface will result in jetting, emission or other instabilities. We stress that Eq. (20) describes a "local" stability condition, so that initiation of instability or breakdown need not occur uniformly over the whole surface. However, once breakdown has occurred dynamic processes such as space charge, excitation of surface capillary waves /17/ etc., can be instrumental in propagating and/or suppressing instability over the rest of the surface. This is to be contrasted with the physical implication of the Taylor stress condition applied to the conical model. As we have previously observed, that equation is interpreted as a stability condition that is independent of position on the cone and it predicts breakdown over the entire surface surface of the fluid at the same voltage.

As an application of Eq. (20) we consider now the determination of the onset, or critical, voltage. As a first approximation, we retain only first-order terms in \( \theta_0 \). Therefore, just prior to onset of the instability, Eq. (20) is

\[
- \frac{1}{2} \left( -p_1 + p_2 \right) - \left( \gamma - \frac{1}{2} V_0^2 \theta_0 \right) \frac{1}{R_1} + \frac{1}{R_2} \left[ 1 - \frac{R_1 R_2}{(R_1 + R_2)^2} \right] \approx 0 .
\] (22)

It is seen that the critical voltage is a function of the curvature via \( R_1 \) and \( R_2 \). \( V_0^C \) also depends explicitly on \( \theta_0 \), which means it implicitly depends in a point-to-point fashion upon the shape of the entire surface. This suggests that the experimental determinations of \( V_0 \) may not always be unique since breakdown is a function of both local configuration and approach to the critical voltage (e.g., experimental arrangement of electrodes, which, in turn, determine the allowed equilibrium configuration of the fluid surface).

Finally, the instability conditions given in Eq. (20) and (21) can be used to rule out certain surface structures as allowed shapes.

VI. SUMMARY

In this paper we have used a set of variational equations, based on the principle of minimum energy, to determine if the cuspidal model is an allowed equilibrium configuration of a stressed conducting fluid in an electric field prior to onset of hydrodynamic instability. It was found that the singularity at the apex of the ideal cusp model introduced mathematical difficulties which suggested that the pointed apex in the cusp model is not compatible with a static equilibrium shape. We also discussed newly derived equations for equilibrium and stability of conducting fluids in an electric field. These were obtained using an analysis based on the virtual-work principle. In interpreting both equations, it is found that the initiation of breakdown need not occur uniformly over the entire fluid surface. This is contrasted with the predictions of the Taylor stress condition for the conical model which predicts breakdown of the entire fluid surface at the same voltage.

+This work was supported in part by the Division of Materials Research, National Science Foundation, Grant No. DMR-8108829.

++Department of Physics, The Pennsylvania State University, Altoona Campus, Altoona, Pennsylvania 16603.

REFERENCES


11. The general features are also suggested in experimental observations of fluid interfaces stressed by a normal electric field. See


