MOMENT MAPS, SYMPLECTIC MANIFOLDS, COHERENT STATES APPLIED TO NUCLEAR COLLECTIVE MOTION

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1 - Introduction

The theory of collective motion in nuclei has as its geometric origin the comparison of certain nuclear phenomena with properties of a liquid drop. Bohr and Mottelson (1952,53) /1,2/ introduced the idea of an irrotational flow and explained a variety of collective phenomena by the deformations and vibrations of a nuclear fluid. The nuclear shell theory succeeded later on in the representation of collective excitation by coherent superpositions of many single-particle excitations. Elliott (1958) /3/ showed that collective levels in the shell theory can be connected to a group SU(3). Independently of the shell theory, various attempts were made to develop the geometric ideas implicit in the Bohr-Mottelson model. Weaver, Biedenharn and Cusson /4,5,6/ introduced the group SL(3, IR) of volume-preserving deformations into the collective theory. With this group, they connected kinematical transformations of the system of A nucleons, the vortex spin, and a spectrum generating algebra. Inclusion of the mass quadrupole tensor leads to a natural extension of this group which was also studied by Rowe, Rosensteel and collaborators /7,8/. In the geometric models, it is the final goal to explain collective phenomena from the point of view of many-body dynamics. Therefore one has to link the collective coordinates to the single-particle coordinates. This program was already started by Lipkin (1955) /9/ and by Villars (1957) /10/. Whereas these authors tried to keep the single-particle coordinates, new viewpoints were developed later by Zickendraht /11/, by Dzyublik et al. /12/, and by Buck Biedenharn and Cusson /13/ by use of orthogonal intrinsic group SO(n, IR) acting on the particle indices and commuting with SO(3, IR). Rowe and Rosensteel (1980) /14/ were the first to analyze this scheme through an orbit analysis in configuration space. Vanagas (1977) /15/ pointed out the close relation of the group SO(n, IR) to the symmetric group of orbital permutations and proposed the group SO(n, IR) as a symmetry group of the collective hamiltonian. In the following sections, the dynamical implications of this proposal will be analyzed, based on work done with Z. Papadopulos, M. Saraceno and W. Schweizer.

Before starting the next section it should be mentioned that the symplectic group Sp(6, IR) or rather Sp(2r, IR) (r = 1,2,3 is the dimension of the space) as a phenomenological group for collective motion was proposed by Goshen and Lipkin for r = 1 /16/ and 2 /17/ and studied by Rosensteel and Rowe /18/ for r = 3.

2 - Group action on many-body phase space, the moment map

Many-body phase space is a manifold $M$ with a Poisson bracket $\{,\}_P$, i.e. a symplectic manifold. As a chart of $M$ we use $n = A-1$ relative Jacobi coordinates $\xi_{is}$ and
momenta $\pi_{is}$, $i = 1,2,3$, $s = 1,2,\ldots, n$. Associate to a point $p \in M$ the $6 \times n$ matrix

$p : \begin{cases} i = 1,2,3 & p_{is} = \xi_{is} \\ i = 4,5,6 & p_{is} = -\pi_{i-3s} \end{cases}$

The fundamental Poisson brackets become

$$\{p_{is'}, p_{kt}\} = K_{lk} \delta_{st'}$$

The group $G = SO(n, \mathbb{R})$ of real orthogonal matrices $g''$, $g''^t g'' = I$ acts on $M$ by

$$G \times M \to M : (g'', p) \to pg'' .$$

The symplectic group $G = Sp(6, \mathbb{R})$ of real $6 \times 6$ matrices $g'$ obeying

$$g' K t g' = K$$

acts on $M$ by

$$G \times M \to M : (g', p) \to (g')^{-1} p .$$

The groups $G$ and $G$ define linear canonical transformations of the observables, that is, on the functions on $M$, through the prescriptions

$$g'' \in SO(n, \mathbb{R}) : (T_{g''} f)(p) = f(p g'')$$
$$g' \in Sp(6, \mathbb{R}) : (T_{g'} f)(p) = f((g')^{-1} p)$$

These two actions commute, that is,

$$(T_{g''} T_{g'} f)(p) = (T_{g'} T_{g''} f)(p) .$$

For $SO(n, \mathbb{R})$ we define now the $n \times n$ matrix

$$T^t g K p ,$$
$$T_{st}(p) = - \sum_{i=1}^3 (\xi_{is} \xi_{it} - \xi_{it} \xi_{is})$$

Then the infinitesimal (Lie-) action of $G$ on $M$ is generated by the Poisson action

$$f(p g'') = f(p) - \sum_{t,s} \alpha^t T_{st} f \{p, \xi_{is}\} + \ldots$$

where $t \alpha = -\alpha$. The formal setting now employs

2.1 Definition: Let $G$ act on $M$ by Poisson action. The functions on $M$ corresponding to the generators of $G$ yield the moment map $m$ from $M$ to $L^*$. The moment map for $G = SO(n, \mathbb{R})$ becomes

$$m : M \to so^*(n, \mathbb{R}) : p \to T(p) = t^p K p$$

where a point of $L^* = so^*(n, \mathbb{R})$ is described by the elements of $T$. There is a natural action of $G$ on $L^*$ called the co-adjoint action $Ad^*$. In the present case it is given by

$$Ad^* g'' (T) = t^g'' T g''$$

For the $G$-actions on $M$ and $L^*$ one verifies the commutative diagram
Corresponding constructions apply to the group \( G = \text{Sp}(6, \mathbb{R}) \). Define the \( 6 \times 6 \) matrix 
\[
T_1(p) = t_p K
\]
The infinitesimal action of \( \text{Sp}(6, \mathbb{R}) \) is generated by the Poisson action 
\[
f((g')^{-1} p) = f(p) - \frac{1}{3} \sum_{i,j} \alpha_{ij} T_{ij} f_p (p) + ... 
\]
The moment map for \( G = \text{Sp}(6, \mathbb{R}) \) is now 
\[
m : M \rightarrow \text{sp}(6, \mathbb{R}) : p \rightarrow T_1(p) = t_p K
\]
The co-adjoint action \( \text{Ad}^* \) on \( L^* = \text{sp}(6, \mathbb{R}) \) is now given by 
\[
(\text{Ad}_{g^*}^*) (T') = (g')^{-1} T' g'
\]
Under the co-adjoint action \( \text{Ad}^* \), the dual \( L^* \) can be decomposed into orbits. These orbits have a natural generalized Poisson bracket, i.e. they are symplectic manifolds. The linear functions on \( L^* \) restricted to these orbits become the generalized Poisson generators for the coadjoint action of \( G \) on \( L^* \), see Arnold /20/.

3 - The reduced phase space for a hamiltonian invariant under \( \text{SO}(n, \mathbb{R}) \)

We shall adopt the characterization of the collective hamiltonian \( H_{\text{coll}} \) due to Vanagas /15/ that \( H_{\text{coll}} \) be invariant under \( G = \text{SO}(n, \mathbb{R}) \). The consequences of this idea are described in the following sections. The analysis of Marsden and Weinstein /21/, see Arnold /20/, shows that for a hamiltonian with symmetry group \( G \) there exists a reduced phase space \( F \) with a generalized Poisson bracket \( \{,\} \). To construct \( F \) we note that the moments \( m(p) \) of \( G \) have vanishing Poisson brackets with \( H = H_{\text{coll}} \) so that fixed values of \( T = (T_{st}) \) can be chosen as integrals of motion. For fixed \( T \), the inverse images of \( T \) under \( m \) form a subset \( M_T = m^{-1}(T) \) of \( M \). The stability group \( H_T \) of \( T \) on \( L^* \) acts on \( M_T \). The reduced phase space is the projection \( F_T = \pi(M_T) \) obtained by factorizing out the action of \( H_T \) from \( M_T \). This subset \( F_T \) is shown to carry under appropriate condition a non-degenerate symplectic structure and generalized Poisson bracket /20,21/.

To implement the reduced phase space, it proves useful to construct the \( G \)-orbits on \( L^* \) and on \( M \) and to relate them. From its definition, the reduced phase space contains points from the transversal to the \( G \)-action on \( M \) which are mapped into the same \( G \)-orbit on \( L^* \). To analyze these transversals we now introduce the collective group \( G = \text{Sp}(6, \mathbb{R}) \) whose elements \( g' \) act on \( M \) according to 
\[
g' \in G : p \rightarrow (g')^{-1} p.
\]
We shall also employ a complex setting defined by 
\[
p_c = R p = \begin{bmatrix} z_1 s \\ z_2 js \\ \end{bmatrix}, \quad R = \sqrt{2} \begin{bmatrix} 1 & iI \\ iI & 1 - iI \\ \end{bmatrix},
\]
and the corresponding complex form of \( G \), 
\[
R \text{Sp}(6, \mathbb{R}) R^{-1} = \text{Sp}(6, \mathbb{C}) \cap \text{U}(3,3)
\]
For \( G \) we get a moment map \( m : G \rightarrow L^* \), \( L = \text{sp}(6, \mathbb{R}) \) as
The 6 x 6 matrices $T'$ and $T$ have the 3 x 3 block form

$$n = \begin{bmatrix} A & 0 \\ -P & t_A \end{bmatrix}$$

$$C_{ij} = \frac{1}{2} \sum_{s=1}^{n} (z_{is} z_{js} + z_{js} z_{is}),$$

$$K_{+,ij} = \frac{1}{2} \sum_{s=1}^{n} z_{is} z_{js},$$

$$K_{-,ij} = \frac{1}{2} \sum_{s=1}^{n} z_{is} z_{js}.$$
For different degeneracies of the matrix \( \sigma \), we obtain the following stability groups \( H \) and dimensions of the cosetspace \( \text{Sp}(6, \mathbb{R})/H \):

\[
\begin{align*}
\sigma_1 > \sigma_2 & > \sigma_3 & U(1) \times U(1) & \times U(1) & 18 \\
\sigma_1 > \sigma_2 & = \sigma_3 & U(1) \times U(2) & 16 \\
\sigma_1 = \sigma_2 & > \sigma_3 & U(3) & 12 \\
\sigma_1 = \sigma_2 & = \sigma_3 & U(3) & 12
\end{align*}
\]

We consider in more detail the case \( \sigma = \sigma_0 I \). The coset \( \text{Sp}(6, \mathbb{R})/U(3) \) may be parameterized by a symplectic matrix

\[
c = \begin{bmatrix} 1 & -B \\ 0 & 1 \end{bmatrix}, \quad tB = B, \quad I - BB^+ > 0
\]

In these complex parameters the form of \( T \) is

\[
T(B, B^+) = \sigma_0 \begin{pmatrix} (I+BB^+)(I-BB)^{-1} & -2(I-BB)^{-1}B \\ 2B^+(I-BB)^{-1} & -(I-BB)^{-1}(I-BB^+) \end{pmatrix}
\]

The fundamental generalized Poisson brackets become

\[
\{b_{ij}, b_{lk}\} = 0, \quad \{b_{ij}^+, b_{lk}^+\} = 0
\]

\[
\{b_{ij}^+, b_{lk}^+\} = \sigma_0^{-1} \left( (I-BB^+)^{i_1}_j (I-BB^+)^{k_1}_l + (I-BB^+)^{k_1}_l (I-BB^+)^{i_1}_j \right)
\]

Under \( U(3) \), the elements of \( B \) transform according to the irreducible representation \( D(2) \) and hence represent an s- and five d-quasi-particles. The interpretation of this orbits in collective theory is obtained through dequantization /23, 26/. Then the degenerate matrix \( \sigma \) can be shown to correspond to the collective dynamics of closed oscillator shell configurations, and \( \sigma_0 \) is given by

\[
\sigma_0 = M/3 + n/2
\]

where \( M \) is the total excitation. For large mass number \( A \), the quasi-particles may be transformed into bosons, and the dynamics on these orbits is given by the interaction of these bosons /27/.

For open-shell dynamics we have additional degrees of freedom which are under study.

5 - Conal orbits and irrotational flow, new approach to hyperbolic orbits

If the condition \( MT > 0 \) is relaxed to \( MT \geq 0 \), there appears the possibility of Jordan decompositions of \( T \), compare Brunet and Kramer /25/. The Jordan chains have a maximum length 2, and if there are three such chains, the standard form of \( T' \) is

\[
\tilde{T}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

We call the orbits with Jordan chains conal, again in analogy to the case of \( SU(1,1) \).

In this case we use the real setting and get as the stability group the group

\[
H = t(6) \times SO(3, \mathbb{R})
\]

with elements
\[
\begin{bmatrix}
1 & \gamma \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 \\
o & a
\end{bmatrix},
\quad t_\gamma = \gamma, \quad t_a = a^{-1}
\]
and $\gamma$ parameters. The coset $Sp(6, \mathbb{R})/H$ admits the parametrization
\[
c = \begin{bmatrix}
I & 0 \\
-Z & I
\end{bmatrix}
\begin{bmatrix}
s & 0 \\
o & s^{-1}
\end{bmatrix},
\quad s = t_s \geq 0, \quad Z = t_Z
\]
The form of $T'$ becomes
\[
T'(s^2, Z) =
\begin{bmatrix}
s^2Z & s^2 \\
-s^2Zs^2 & -zs^2
\end{bmatrix}
\]
and the fundamental generalized Poisson brackets are
\[
\{ (s^2)_{ij}, (s^2)_{lk} \} = 0, \quad \{ Z_{ij}, Z_{lk} \} = 0,
\]
\[
\{ (s^2)_{ij}, Z_{lk} \} = \delta_{i1} \delta_{jk} + \delta_{ik} \delta_{jl}
\]
Since $s^2 = Q$, the matrix $Z$ determines generalized momenta corresponding to $Q$. If now one computes the positions and momenta on this orbit one finds
\[
\pi_{is} = \frac{3}{2} Z_{ij} \varepsilon_{js} \quad s = 1, 2, \ldots, n
\]
Since $Z = tZ$, the momenta and (for local interactions) the velocities are linearly related to the positions, and the velocity field is irrotational as postulated by Bohr and Mottelson [1, 2].
A non-trivial extension of these coordinates to hyperbolic orbits is obtained by writing
\[
T' = \sigma_0
\begin{bmatrix}
s^2Z & s^2 \\
-s^2Zs^2 & -zs^2
\end{bmatrix}
\]
and the replacement $s^2 \rightarrow \omega_0 s^2$ in the Poisson brackets. Then $Q$ and $Z$ remain canonically conjugate. Passing from $Q$ and $Z$ to three angular coordinates $q_\ell$ and three eigenvalues $\omega_\ell$ of $Q$ and to canonical momenta $\pi_\ell$ and $\pi_1$ respectively, one obtains for the collective hamiltonian
\[
H_{coll} = \pi_\ell \frac{1}{2} \frac{3}{\ell} 4\sum_{r=1}^{3} (\omega_r)^{-1} + \pi_\ell \frac{1}{2} \frac{3}{\ell} \sum_{j=1}^{3} J_j^{-1} (L_j)^2
\]
\[
+ \pi_\ell \frac{1}{2} \sum_{r=1}^{3} (\omega_r)^{-1} + \sum_{r=1}^{3} (\omega_1 + \omega_2 + \omega_3)
\]
\[
J_j = (c_{jk1})^2 (\omega_1 - \omega_k)^2 (\omega_1 + \omega_k)^{-1}
\]
where $L^b_j$ is the angular momentum referred to the body-fixed system. The study of the classical dynamics for this hamiltonian exhibits the relation between the Pauli principle contained in the constant $\omega_0$, the angular momentum and the interaction, compare unpublished work by P. Kramer, Z. Papadopolos and W. Schweizer.

6 - Coherent states and dequantization

Up to now, a classical scheme for nuclear collective model has been discussed. This classical scheme can be connected to a quantum scheme through the concept of coherent states. For the classical scheme, the collective phase space was found to be a coset $Sp(6, \mathbb{R})/H$ of the symplectic group. Denote again the generators of this coset by $c$ and consider in the many-body state space the coherent states
\[
\langle c | \psi \rangle = \langle \text{extr} | U(c) | \psi \rangle
\]
Here, $\langle \text{extr} \rangle$ is an extremal state for the representation of the group $\text{Sp}(6, \mathbb{R})$. In many cases, these coherent states belong to a Hilbert space with a measure with respect to the coset $\text{Sp}(6, \mathbb{R})/H$. Then the collective dynamics can be formulated on this collective state space, compare [23].

The classical approximation is constructed in the following manner: To a quantum observable $X$ we associate a classical observable on $\text{Sp}(6, \mathbb{R})/H$ by

$$X = \langle c | X | c \rangle / \langle c | c \rangle$$

Let $\xi_1, \xi_2, \ldots, \xi_r$ be the real parameters of the coset generators $c$, 

$$c = c (\xi_1, \xi_2, \ldots, \xi_r)$$

and define the matrix $C$ with elements

$$C_{\alpha\beta} = (\delta^2 / \delta \xi^\alpha \delta \xi^\beta) \ln \langle c(\xi') | c(\xi) \rangle$$

Then the generalized Poisson bracket $\{,\}$ given by

$$\{f, g\} = \sum_{\alpha, \beta} (\partial f / \partial \xi^\alpha) C^{-1}_{\alpha\beta} (\partial g / \partial \xi^\beta)$$

yields a Poisson realization of the Lie algebra of the group, in this case $\text{Sp}(6, \mathbb{R})$, whose classical generators are obtained as the expectation values of the quantum generators. The general construction is given in [26], the specific constructions for collective theory are given in [22, 23]. They yield as generalized Poisson brackets precisely the ones discussed in previous sections. In addition they determine the values of the constants appearing in the classical expressions and hence yield the appropriate interpretation of the classical scheme. With respect to the quantum system, the classical approximation implies that we adopt a time-evolution where, the state is a coherent wave packet at any time.

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References