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THE GREEN FUNCTION OF A MACROSCOPIC ELECTROMAGNETIC FIELD IN A DISPERSIVE PLANE-STRATIFIED ANISOTROPIC MEDIUM

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Résumé - La méthode des équations fonctionnelles développée auparavant dans la construction de la fonction dynamique Green d'un champ quasi-élastique et d'un champ scalaire, dans un milieu dissipatif, anisotrope, stratifié, est étendue à un champ électromagnétique dans un milieu analogue.

Abstract - The method of functional equations, which was developed previously to construct the dynamic Green function of a quasi-elastic field and of a scalar field, in a dissipative plane-stratified anisotropic medium, is extended to an electromagnetic field in an analogous medium.

In this letter, we show that, in the absence of an external static magnetic field, the Green function problem for an electromagnetic field in a dispersive (dissipative) plane-stratified anisotropic medium is reduced to the same chainlike set of functional equations of the Dyson type, which was established and solved previously /1-3/ for the dynamic Green function of a quasi-elastic and of a scalar field, in a similar medium.

In the $\omega$-representation, Maxwell's equations for the macroscopic electromagnetic field which is produced in any medium by the extraneous sources of the charge density $\rho_{\text{ex}}(r;\omega)$ and of the current density $j_{\text{ex}}(r;\omega)$ can be written, using Gaussian units,

$$
e^{ijk}b^k = c^{-1}[-i\omega e^{ij} + 4\pi(j_{\text{in}}^i + j_{\text{ex}}^i)] /a/, \quad \nabla \cdot B = 0 /b/,$$

$$
e^{ijk}e^k = ic^{-1}\omega e^{ij} /c/, \quad \nabla \times E = 4\pi(\rho_{\text{in}} + \rho_{\text{ex}}) /d/,$$

where $E = E(r;\omega)$ is the macroscopic electric field strength, $B = B(r;\omega)$ is the macroscopic magnetic field strength (traditionally called the magnetic induction); $\rho_{\text{in}} = \rho_{\text{in}}(r;\omega)$ is the macroscopic density of intrinsic charges in the medium; $j_{\text{in}} = j_{\text{in}}(r;\omega)$ is the macroscopic density vector of intrinsic currents in the medium; $e^{ijk}$ is the completely antisymmetric unit pseudotensor; Latin superscripts denote components of a vector or of a tensor in an arbitrary Cartesian coordinate system, which is at rest relative to the medium; $r=(r_1,r_2,r_3)=(x,y,z)$ is the position vector of a variable space point with respect to that coordinate system; $\omega$ is a variable frequency; $c$ is the speed of light in vacuum. Each Latin superscript assumes the values 1,2,3 (or $x,y,z$). Correspondingly, a repeated Latin superscript assumes the values 1,2,3. The $t$-representation $\psi(r,t)$ and the $\omega$-representation $\psi(r;\omega)$, of a quantity $\psi$ are related by

$$
\psi(r,t) = \frac{1}{2\pi} \int_0^\infty \psid(r;\omega)e^{-i\omega t}d\omega /a/, \quad \psi(r;\omega) = \frac{1}{2\pi} \int_0^\infty \psitd(r,t)e^{i\omega t}dt /b/.
$$

where $t$ is the time according to a clock which is at rest relative to the medium. We suppose that the extraneous charges and currents do not belong to the given medium, i.e. they are brought in from outside. Accordingly, we suppose that, firstly, $\rho_{\text{ex}}(r;\omega)$ and $j_{\text{ex}}(r;\omega)$ are given (known, prescribed) functions of $r$ and $\omega$, and that, secondly, each pair $\rho_{\text{ex}}$ and $j_{\text{ex}}$, and $\rho_{\text{in}}$ and $j_{\text{in}}$ satisfies an independent continuity...
which are the conservation laws for the extraneous and for the intrinsic charges, respectively. Equation (3a) is postulated. Equation (3b) can be derived by applying the operator \( -i\omega^{-1}V \) to both sides of eq. (1a) and by using eqs. (1d) and (3a). The spectral composition of the extraneous currents is supposed to be such that

\[
\lim_{\omega \to 0} \left| \omega^{-1}V_j^{(e)}(r;\omega) \right| = 0, \quad \lim_{\omega \to 0} \left| \omega^{-1}V_j^{(i)}(r;\omega) \right| = 0.
\]

Equation (4b) follows from eq. (4a), if non-linear effects for the electromagnetic field in question are negligible.

The material equation for \( \mathbf{j}^i \) is supposed to have the form

\[
\mathbf{j}^i_{\text{in}} = \xi \mathbf{j}^i_{\text{m}} \quad /a/, \quad \mathbf{j}^i_{\text{m}} = \varepsilon^{ij} \mathbf{V}^j \mathbf{M}^k \quad /b/,
\]

where

\[
\mathbf{j}^i_{\text{e}}(r;\omega) = \varepsilon^{ij} \mathbf{E}^j(\mathbf{r};\omega) \quad /a/, \quad \mathbf{M}^i(\mathbf{r};\omega) = \chi^{ij} \mathbf{B}^j(\mathbf{r};\omega) \quad /b/;
\]

\( \mathbf{j}^i \) is the density vector of the intrinsic chargeless current; \( \mathbf{M}^i \) is the magnetic moment of the medium per unit volume; \( \varepsilon^{ij} \) is the conductivity tensor; \( \chi^{ij} \) is the magnetic susceptibility tensor; all the above quantities are taken in the \( \omega \)-representation, in accordance with eqs. (2). The fact that \( \varepsilon^{ij} \) and \( \chi^{ij} \) are supposed to depend on \( \mathbf{r} \), signifies that, in general, the medium under consideration can be spatially inhomogeneous. Also, we suppose that \( \varepsilon^{ij}(\mathbf{r};\omega) \) and \( \chi^{ij}(\mathbf{r};\omega) \), when regarded as functions of the complex variable \( \omega \), have no poles in the upper semi-plane. Substituting \( \varepsilon^{ij}(\mathbf{r};\omega) \) and \( \chi^{ij}(\mathbf{r};\omega) \) in turns for \( \psi(\mathbf{r};\omega) \) in eq. (2a), we obtain \( \varepsilon^{ij}(\mathbf{r},t) = \chi^{ij}(\mathbf{r},t) = 0 \) for \( t < 0 \). As a consequence, in the \( t \)-representation eqs. (6) become

\[
\mathbf{j}^i(\mathbf{r},t) = \int_0^t \delta^{ij}(\mathbf{r},t') \mathbf{E}^j(\mathbf{r},t-t') \, dt', \quad \mathbf{M}^i(\mathbf{r},t) = \int_0^t \delta^{ij}(\mathbf{r},t') \mathbf{B}^j(\mathbf{r},t-t') \, dt'.
\]

Thus, eqs. (6a) and (6b) subject to the above assumptions are the most general linear cause-and-effect relations between \( \mathbf{E} \) and \( \mathbf{J}^i \), and between \( \mathbf{M}^i \) and \( \mathbf{B}^j \), respectively, which allow for the frequency dispersion of the medium, but which leave out of account a possible space dispersion of the medium. A macroscopic chargeless current is associated with the corresponding microscopic spin current, and is, hence, essential only if the medium has a magnetic structure, i.e. either a ferromagnet or an antiferromagnet. If the medium has no magnetic structure, we can set \( \chi^{ij} = 0 \), thus neglecting \( |\mathbf{M}^i| \) as compared to \( |\mathbf{J}^i_m| \).

Substitution of eqs. (5) into eq. (1a), and of eq. (3b) into eq. (1d) yields

\[
\varepsilon^{ij} \mathbf{V}^j \mathbf{H}^k = c^{-1} (-i\omega \delta^{ij} + 4\pi \varepsilon_{\text{ex}}^i)^k /a/, \quad \mathbf{V}^k \mathbf{D}^k = 4\pi \rho_{\text{ex}} /b/,
\]

where, by definition,

\[
\mathbf{D}^i = \mathbf{E}^i + 4\pi \mathbf{j}^i_{\text{ex}} /a/, \quad \mathbf{H}^i = \boldsymbol{\mu}^i_{\text{ex}} /b/;
\]

\( \mathbf{D}^i \) is the vector of electric induction; \( \mathbf{H}^i \) is the vector of magnetic induction (traditionally called the magnetic field strength). In view of eqs. (6), eqs. (8) become

\[
\delta^{ij} = \varepsilon^{ij} \mathbf{E}^j /a/, \quad \mathbf{H}^i = \eta^{ij} \mathbf{B}^j /b/;
\]

\( \varepsilon^{ij} \) is the dielectric permeability tensor; \( \eta^{ij} \) is the inverse of the magnetic permeability tensor \( \mu^{ij} \). Denoting, also, the inverse of \( \varepsilon^{ij} \) by \( \xi^{ij} \), we have, by definition,

\[
\xi^{ij}(\mathbf{r};\omega) = [\varepsilon^{-1}(\mathbf{r};\omega)]^{ij} /a/, \quad \mu^{ij}(\mathbf{r};\omega) = [\eta^{-1}(\mathbf{r};\omega)]^{ij} /b/.
\]
According to the definition of $\sigma_{ij}(\vec{r};\omega)$ and of $\chi_{ij}(\vec{r};\omega)$, the quantities $\epsilon_{ij}(\vec{r};\omega)$ and $\eta_{ij}(\vec{r};\omega)$ have no poles in the upper semi-plane of the complex variable $\omega$. Note that usually the tensor $\mu_{ij}(\vec{r};\omega)$, and not $\eta_{ij}(\vec{r};\omega)$, is supposed to be regular in the upper semi-plane of $\omega$.

Multiplying eq. (1c) by $\eta_{mn}$, and then making use of eq.(9b), we have

$$H^m = -i\omega^{-1} \eta_{mn} e^{nk} \nu^k \nu e. \tag{12}$$

If we insert eqs.(9a) and (12) into eq.(7a), we obtain

$$K^i_k(\vec{r};-iV,\omega)E^o_{k}(\vec{r};\omega) = -4\pi i\omega \epsilon_{ex}^i(\vec{r};\omega), \tag{13}$$

$$K^i_k(\vec{r};-iV,\omega) = \omega^2 \epsilon_{ex}^i(\vec{r};\omega) + \nu^l \gamma^i_j k^j \nu^k(\vec{r};\omega) \nu^l, \tag{14}$$

$$\gamma^i_j k^j \nu^k(\vec{r};\omega) = e^{ijm} \eta_{mn}(\vec{r};\omega) \nu e. \tag{15}$$

A similar equation can be derived for $\vec{H}$. To this end, we insert eq.(9a) into eq.(7a), and we multiply the equation so obtained by the tensor $\xi^m_i$, eq.(11a). This gives

$$E^m = i\omega^{-1} \xi_{mn} (ce^{jk} \nu H - 4\pi j n). \tag{16}$$

Applying the operator $e^{ijm} \gamma^j \nu$ to both sides of eq.(16), and then making use of eqs.(10) (9b), and (11b), we obtain

$$K^i_k(\vec{r};-iV,\omega)H^o_{k}(\vec{r};\omega) = \phi^i(\vec{r};\omega), \tag{17}$$

$$K^i_k(\vec{r};-iV,\omega) = \omega^2 \mu_{ex}^i(\vec{r};\omega) + \nu^l \gamma^i_j k^j \nu^k(\vec{r};\omega) \nu^l, \tag{18}$$

$$\gamma^i_j k^j \nu^k(\vec{r};\omega) = e^{ijm} \xi_{mn}(\vec{r};\omega) \nu e, \tag{19}$$

$$\phi^i(\vec{r};\omega) = -4\pi \epsilon_{ex} \gamma^i_j \nu^j \xi_{mn}(\vec{r};\omega) \nu e(\vec{r};\omega). \tag{20}$$

We define the electromagnetic Green function $E^{o}_{ii}(\vec{r},\vec{r}_0;\omega)$ of the $E$-type and the electromagnetic Green function $H^{o}_{ii}(\vec{r},\vec{r}_0;\omega)$ of the $H$-type as the solutions of two independent equations:

$$K^i_k(\vec{r};-iV,\omega)E^{o}_{k}(\vec{r},\vec{r}_0;\omega) = \delta(\vec{r}-\vec{r}_0) \delta^i, \tag{21}$$

$$K^i_k(\vec{r};-iV,\omega)H^{o}_{k}(\vec{r},\vec{r}_0;\omega) = \delta(\vec{r}-\vec{r}_0) \delta^i, \tag{22}$$

in the whole infinite space, where $\vec{r}_0$ and $i_0$ are parameters ($i_0 = 1,2,3$).

Since eqs.(13) and (17) are linear, their solutions can be written

$$E^i(\vec{r};\omega) = -4\pi i\omega \epsilon^i(\vec{r},\vec{r}_0;\omega)_{ex} \delta(\vec{r}-\vec{r}_0) d^3 \vec{r}, \tag{23}$$

$$H^i(\vec{r};\omega) = \delta(\vec{r},\vec{r}_0;\omega) \phi^i(\vec{r};\omega) d^3 \vec{r}. \tag{24}$$

The vectors $D^i$ and $B^i$ can then be found by eqs.(9) and (11b). Substitution of eq.(23) into eq.(12), and of eq.(24) into eq.(16) enables us to express $H^i$ in terms of $E^{ii}_o$, and $E^i$ in terms of $H^{ii}_o$. Hence, only one of the two Green functions is actually required.

We rewrite eqs.(21) and (22) as one equation of the form

$$K^i_k(\vec{r};-iV,\omega)G^{o}_{k}(\vec{r},\vec{r}_0;\omega) = \delta(\vec{r}-\vec{r}_0) \delta^i, \tag{25}$$

where

$$K^i_k(\vec{r};-iV,\omega) = \omega^2 \alpha_{ex}^i(\vec{r};\omega) + \nu^l \gamma^i_j k^j \nu^k(\vec{r};\omega) \nu^l; \tag{26}$$
In the absence of a static external magnetic field, the tensors $\alpha^i_k$ and $\beta^i_k$ are symmetric. Since the tensor $\varepsilon_{iik}$ is antisymmetric with respect to the permutation of any two indices, we have

$$Z^{ijkk} = Z^{kiij} = Z^{ijkj} = Z^{jikj}.$$  

We observe that eq.(25) subject to eq.(26) turns into the equation for the acoustic vibration field Green function $D^{\mu}_{ij}(r,r';\omega)$ of a spatially inhomogeneous elasto-viscous medium with the mass density $\rho(r)$ and with the complex elasticity tensor $C^{ijk}\rho_{ij}(\omega)$ (see refs./1,2/) if in eq.(26) we formally set

$$G^\mu_0 = g^\mu_0, \quad \alpha^i_k = \rho(r) \delta^i_k, \quad Z^{ijk}_{\mu} = Z^{ijk}_{\mu}(r;\omega).$$  

In this case, instead of eqs.(32), we have

$$Z^{ijkk} = Z^{kiij} = Z^{ijkj} = Z^{jikj}.$$  

Suppose now that the medium under consideration consists of $n+1$ homogeneous anisotropic plane-parallel layers of arbitrary thicknesses in contact. We align the axis $x$ with a normal to the interfaces, and denote by $\alpha^i_k\beta^j_l(\omega)$, $Z^{ijk}_{\mu}(\omega)$ the corresponding material characteristics of the individual layers; $\mu = 0, \ldots, n$. In this case, we may write

$$G^\mu_0 (r,r';\omega) = (2\pi)^{-2} \int G(x,x';\omega) e^{i\vec{k}\cdot\vec{r}} d^2 k, \quad \vec{k} = (0,k_y,k_z), \quad k = (0,k_y,k_z),$$  

the variable $\omega$ being omitted. The quantity $G^\mu_0 (x,x';\omega)$ is called the $(x,\vec{k})$-representation, or in words, the mixed coordinate-propagation-vector-representation, of the total Green function. If $\alpha^i_k = \alpha^i_k(\omega)$ and $Z^{ijk}_{\mu} = Z^{ijk}_{\mu}(\omega)$ in the whole infinite space, the corresponding solution of eq.(25) is denoted by $G^\mu_0 (r,r')$ and is called the standard Green function of the $\mu$th kind. The $(x,\vec{k})$-representation $G^{\mu}_{\vec{k}}(x-x';\omega)$ of this function is the solution of the equation

$$[\omega^2 \alpha^i_k(\omega) + Z^{ijk}_{\mu}(\omega) \delta^i_j] G^{\mu}_{\vec{k}}(x) = \delta(x) \delta^{\vec{k}}, \quad \alpha^i_k = \delta^i_j \frac{\partial^2}{\partial x^i} \delta^{\vec{k}}, \quad -\infty < x < \infty.$$  

From here on the variable $\vec{k}$ of every function in the $(x,\vec{k})$-representation will be omitted.

In refs./1,2/, we showed that the acoustic vibration field Green function in the $(x,\vec{k})$-representation, of the plane-stratified medium as mentioned, satisfies a chainlike set of $n+1$ 3x3-matrix functional equations of the bifurc type, which include both the equation of motion and the corresponding boundary conditions at each interface. In deriving that set of equations, it was essential that the elasticity tensor was invariant under the permutation $ij=k\ell$ (see eqs.(30) and (31)). Since the material tensor $Z^{ijk}_{\mu}$ as defined by eq.(27) possesses the same symmetry property (see eqs.(29)), the set of functional equations of refs./1,2/ remains valid also in the case of an electromagnetic field. One should only substitute $G^{\mu}_{ij}(x,x')$ for $D^{\mu}_{ij}(x,x')$, and $G^{\mu}_{\omega}(x)\delta(x)$ for $D^{\mu}_{\omega}(x)\delta(x)$ ($\omega = 0, 1, \ldots, n$).

Thus, we have the following set of equations for $G^{\mu}_{ij}(x,x')$:

$$\theta(\delta - 0) G^{\mu}_{ij}(x,x') = 0 (\delta - 0) G^{\mu}_{ij}(x,x'), \quad \mu = 0, \ldots, n.$$  

(34)
The separation plane between the \( \mu \)th and the \((\mu+1)\)th layer. The zeroth \((\mu=0)\) and the last \((\mu=n)\) layer are supposed to be semi-infinite spaces \( x<d_0 \) and \( x>d_{n-1} \), respectively (\( d_{-1}=\infty, d_{n}=\infty \)). Equations (38) are the usual boundary conditions of continuity for the tangential components of \( \hat{E} \) and \( \hat{H} \) (or correspondingly \( \hat{E} \) and \( \hat{H} \)) in the case, when \( \hat{E} \) is the Green function of the \( E \)-type (or correspondingly, when \( \hat{H} \) is the Green function of the \( H \)-type). At the same time, it follows from eqs. (27), (39), and (40) that \( p^{ij}(x, x_0)=0 \) and \( T^{ij}(x)=0 \). In view of the last equation, eq. (37) does not involve \( G^{i0}_{\lambda}(d_\nu \pm 0, x_0) \). These peculiarities of eqs. (34)-(37) reflect the fact that the electromagnetic field is transverse. In this connection, we may notice that eqs. (34)-(37) are, in a sense, complementary to the corresponding equations for the acoustic vibration field Green function in a stratified ideal fluid. The latter field is purely longitudinal. Equations (34)-(36) can be solved largely by the same methods, which were suggested in refs. /1-3/. The Green function theory of electromagnetic fields in stratified media will be discussed in detail in further publications. This theory will, also, include the case, when a stratified medium is in a static external magnetic field.

References